MATHEMATICAL AND METAPHYSICAL LANGUAGES

JAVIER LEACH
Universidad Complutense de Madrid

ABSTRACT: This paper summarizes some of the ideas that I will present in the book *Mathematics and Religion: Our languages of Sign and Symbol*, which will be publish by Templeton Press in the month of September 2010. This book is about our languages, but with a focus on the privileged role that mathematics has in our ability to communicate about the world around us. In this sense, mathematics is our «public language». But it is more than that also. As I hope you will see, mathematics leads us through science and brings us to questions about a greater reality, which we call metaphysical reality and which we usually approach in the context of philosophy and religion.

KEY WORDS: signs, symbols, language, formal sciences, empirical sciences, logic, mathematics, metaphysics, religion, theology.

Lenguajes matemáticos y lenguajes metafísicos

RESUMEN: Este artículo compendia algunas de las ideas que presentaré en el libro de *Matemáticas y Religión: Signos y símbolos*, que publicará la Editorial Templeton Press hacia el mes de septiembre de 2010. Este libro trata del lenguaje en general, pero con un enfoque en el papel privilegiado que la matemática tiene en nuestra capacidad para comunicar las percepciones. En este sentido, la matemática será nuestro «lenguaje público». Pero es más que eso también. Como espero que se vea, la matemática nos lleva a través de la ciencia a preguntas acerca de una realidad mayor, que llamamos realidad metafísica y a la que por lo general nos acercamos en el contexto de la filosofía y la religión.

PALABRAS CLAVE: signos, símbolos, lenguaje, ciencias formales, ciencias empíricas, lógica, matemáticas, metafísica, teología.

The first chapters of the book will focus on defining and explaining the three basic languages that concern us. Each has a particular kind of perception of reality and then a system of signs or symbols (a language) to convey that perception.

The first kind of perception is found in logic and mathematics, a purely mental kind of perception. It uses the language of «formal signs». This language often seems to be an array of impenetrable hieroglyphics, such as the notation $\Sigma ar = (0, 1, +, \times, <)$. But as we will see, it is a language that can be understood to some degree by everyone. Of course, it is also a very specialized language: it is the lingua franca of logicians and mathematicians.

The second kind of perception of the world is through empirical science. It uses «representational» language to convey that perception. This language speaks of physical realities such as weight, force, or mass. It is the language of physics, chemistry, geology, or neuroscience. Famously, for example, Newton gave us a representational language to talk about the force of gravity based on the size and distance between objects such as planets. Einstein did likewise with a language that says energy equals mass in the particular arrangement of $E = mc^2$.

Finally, a third kind of human perception and language is metaphysical. It is a logical language just like mathematics, but it uses symbols (not «signs») to explain perceptions of relationships, causes, and ultimate reality. These symbols have included the idea of a transcendent God, a being that surpasses all other beings, or a being who is in relation to the world, but is nevertheless beyond the world. Metaphysical language uses symbols to speak about the individuality and unity of things, the nature of the infinite, the scope of the universe, or the relationships we call community.

It took most of human history for us to arrive at our modern understanding of mathematical language. The largest section of the book will be a survey of the evolution of our mathematical systems. I have found this the best way to introduce mathematics to a general audience. In hindsight, we can now see the turning points in our growing knowledge of mathematics and reflect on the colorful stories of the people who moved the science of mathematics up to its present state. As we will see, that current state is characterized by the «formalization» of mathematical systems.

We will conclude with a look at how we derive meaning from the semantics of our various languages — mathematical, empirical, and religious — and in what ways metaphysical questions have become more important as our culture grows more scientific. Our ability to understand our various kinds of languages will help us not only in taking advantage of a complex scientific world, but in deepening our search for personal answers to the great metaphysical questions.

In the past three decades, the world has seemed to change more rapidly than ever. Math and science have grown in importance. To the surprise of many, we also see a perennial return to religion. In such times, I believe that mathematics retains a privileged position because of its unique role in linking, by the principles of logic, science with philosophy and theology. In making this case, I speak as a Christian who values the interfaith spirit or our age and the age-old tradition of humanist learning.

1. The languages of Mathematics and Natural Sciences

Although mathematics and natural science are closely bound together, they are essentially two different kinds of languages. Mathematics refers primarily to objects of the mind. Natural sciences refer to objects of our sense experience. In mathematics we use abstract «formal» signs (that is, language of precise mental meaning and a language that can be used and manipulated mechanically). In contrast, natural science uses what we may call «representational» language that speaks of the physical objects studied by physics, chemistry, geology, or neuroscience.

While it is widely agreed that language is the means by which we convey meaning, I believe there is too little reflection on how the language of mathematics and the language of religion may share common characteristics. It is too simple to state that they are two alien types of language. Therefore, I offer models for how the languages of science and religion complement each other. Science and religion live in a complex relationship (what I will call an «asymmetrical magisterium»). But they can both offer valid truths based on a common criterion of internal consistency and usefulness in the world.

Formal signs in logic and mathematics

What we perceive at the level of logic is correct reasoning, an inference that one thing naturally leads to another. We can test such logical inference in formal models of logic or we can test them mechanically, as in a computer. But many times we perceive something as logical simply by the power of intuition: it immediately seems to be so. These are logical intuitions. They intuit that something is following the rules of logic. For example, the principle, «It is impossible that something be true and false at the same time», is a logical principle that we intuit is always valid. We call this the principle of non-contradiction.

The logic we intuit can also be put into a formal language. As evidence by the abstract signs often seen in logic or mathematics, the formal language consists of a finite series of signs that follow rules, what we call the rules of syntax. The signs have no definite meaning until they are related to each other by the rule, and then these strings of signs can be interpreted as true or false. That is the language of logic, but it sets the stage for the language of mathematics.

The way to understand their relationship — logic and mathematics — is to say that while mathematics includes logic, it cannot be reduced to formal logic. Mathematics has something more, a kind of mathematical intuition and freedom based on logic. In fact, if we reduce mathematics to pure formal logic we end up with paradoxes, which amount to contradictions. This paradox was illustrated by the great mathematical ambition of the German Gottlob Frege (d. 1925) and the Englishman Bertrand Russell (d. 1970), both of whom wanted to reduce mathematics to formal logic. The result, however — which they conceded — is that such an effort ends in paradoxes. So again, logic and mathematics are different despite many similarities.

Like logic, however, mathematics also begins with intuitive perceptions. Mathematics begins as a purely intellectual exercise driven by intuition. One of the first great mathematicians, Euclid, proposed many of these natural intuitions. For example, the first Euclidian postulate expresses the mathematical intuition that between any two points always a straight line segment can always be drawn. In applying mathematics, we give these intuitions another name: we call them mathematical axioms, which amount to a belief that we presume to be true.

Representational signs in natural science

The natural sciences begin with perceptions of the objects in the world. This is what makes them different from the purely mental starting point of logic and mathematics, though natural science will employ these as well. The empirical observations of natural science can be very sophisticated. Still, they are limited by perceptions of the five senses. Once the perception is made by scientists, they may certainly express it in natural languages, just as Copernicus spoke in Polish or German — and used Latin as academic language — when he talked about his belief that the sun was stationary and the earth moved. But finally, natural science seeks a high precision by the use of applied mathematics. It is here that mathematics becomes a privileged language — scientists understand each other and can conduct identical experiment despite their different national languages.

In practice, of course, the argument that natural science uses «representational signs» differently from the way logic or mathematics uses «formal signs» is a bit more subtle and complicated. The same *signs* can be used in either case. For instance, the signs E, m, and c in the equation $E = mc^2$ can be either formal or representational. As formal signs they stand for elements of a mathematical structure such as the system of real numbers (one of our more complex and inclusive sets of number systems). As representational signs, E, m, and c represent energy, mass and speed of light. The difference between the two uses lies in the fact that, in the former, the semantics refer to mental objects (such as pure numbers) while, in the latter, the semantics refer to physical observations.

But we should emphasize again that logic and mathematics are at the core of natural science. Mathematics is indispensable in scientific research. The instruments that natural science uses to measure physical observations are designed based on mathematical theories. Moreover, logic and mathematics are not merely languages alone. They are the basic logical intuitions that we cannot separate from our empirical sense experiences. And the bridge between those intuitions and our sense experiences has, in the history of science, been the building of scientific models.

Formal and representational models

When we create models, we have structures that help us imagine how the world works. Models are mediators between perception and theories. In science, these models designate and describe the relations between the parts of a given domain of discourse and the procedures we can use to analyze the topic of research.

Naturally, science builds formal models of logic and mathematics and it also builds representational models that describe empirical observations (such as the wooden solar system model of early astronomers). The first (formal) is real, but it is purely conceptual and it does not have to necessarily match reality «out there», for it only has to have internal consistency. A representational model, however, must somehow match the empirical reality that the ordinary person can observe.

The models may be used together, depending on the problem that science is trying to solve. I mentioned earlier the model of non-Euclidian geometry, which essentially speaks of something we call «curved» space, as distinct from the normal «flat» space of geometry. So a model of non-Euclidean geometry can be created to talk about a reality that is not known to us — it is speculative, in this sense. But also, a model of non-Euclidean geometry can also be a representational picture of the physical reality spoken of by Einstein's theory of relativity, which is a mathematical theory that has been verified by the observation of the curvature of light in space.

For another illustration of how these models interact, we can turn again to the story of Nicholas Copernicus in the sixteenth century. In his day, the earth-centered model of Ptolemaic astronomy had dominated Europe for more than a thousand years. It dominated because Ptolemy's representational model was successful in explaining what astronomers saw in the skies year after year. However, Copernicus offered a mathematical model that explained the physical observations just as well — and also more simply than Ptolemy's model.

Let us also consider Einstein's work again. His theory of relativity was a purely mathematical model since he was not an astronomer (and indeed, neither was Copernicus for the most part). Einstein built in his mind a model that tried to create a logical system to explain the universe on scales that were too large to measure physically. Eventually, Einstein's mathematical model was tested in the empirical world. That came in 1919 when the English astronomer Arthur Eddington journeyed to the Island of Principe during a total eclipse to measure, by photography, whether light was bent by curved space as Einstein's model argued. In turn, the successful photographs (rough and questionable as they were) could be explained by Einstein's mathematics. Today, our common scientific language refers to curved space and the four dimensions of space-time: these are representational models based on the formal mathematical concepts.

The lesson here is that in the history of science, empirical observations have usually been interpreted in more than one way. We refine our knowledge by trying to reconcile a mathematical model with a physical model that we can observe. We are most satisfied when this harmonizing of math and observation works very well, but in the mysteries and complexities of the universe, there is no guarantee.

Formalism and objectivity

Although we may affirm that logic and mathematics are the most objective knowledge, they are not totally objective or totally independent of the knowing subject. The view of what is logical and what is mathematical depends on the principles of logic which we accept. There are communities of mathematicians that accept certain logical principles which other communities do not accept.

Even though the logical processes of deduction — with formal syntax and formal semantics — are objective and automatic routines that can be executed by a machine, there are several different possible views of what is logic. Accepting one view or another of logic can depend on personal tastes and preferences for what counts as «valid» logic. One can assume different views of logic, but not at the same time if they are not consistent. Once a view is assumed, consistency must be maintained.

Surprising as it may seem, not all logicians accept the famous principle called «excluded middle», for example. It states that «all propositions are true or false». Classical logic is strongly rooted in the principle of excluded middle. But there are others schools of thought in mathematics, such as the constructivist or intuitionist schools, that do not accept this as an absolute premise. This debate over the excluded middle is a disagreement about the existence of what we call mathematical objects. The classical view is that to prove the existence of a mathematical object it is enough to derive a contradiction from the assumption of its non existence. According to the contrary view (constructivist) it is necessary to find (or construct) any mathematical object in order to prove its existence.

2. Metaphysical Language

If we want to keep our lives as simple as possible, mathematics and natural science offer a great advantage. Neither of them asks ultimate questions. Metaphysics, in contrast, is about ultimate things. For this reason, metaphysical questions may complicate our lives, but they also promise to resolve some of the deepest mysteries. Mathematics and natural science accept reality as a given fact — the questioning ends there. However, metaphysics asks why our minds are able, in fact, to understand the physical world and mathematics. Indeed, metaphysics asks why thing exist at all. Why is there something rather than nothing?

Metaphysics offers a more global and radical way of asking questions about reality. It points to the possible existence of an ultimate principle that justifies the existence of things in general. For most people, the metaphysical questions are unavoidable. But we also know from history that there have been many kinds of perceptions of ultimate things, and therefore many kinds of answers to metaphysical questions. These questions and answers also form distinct communities, such as cultures or religions, just as we have seen in mathematics.

So for example, one group may look at the metaphysical evidence in life and arrive at a basic intuition that the world exists on its own and for its own reasons. We may call this a pantheistic metaphysical view. Another kind of ultimate view may be called agnostic. It argues that we cannot know ultimate principles, so whether they exist or not is irrelevant. Third, a theist may believe that the universe exists because God exists and this Creator has made and maintains the universe. Finally, another metaphysical school may say that the ultimate principle in the universe is mathematics itself, as we shall see with schools such as the Pythagoreans and certain Platonists.

Metaphysics, of course, uses a different language from logic, mathematics, and natural science. This is the language of symbols that stand for ultimate realities or ultimate types of relationships. These symbolic words can range from God and the cosmos or universe to words found in mathematics, if that is deemed the highest reality. Whatever the language-symbol in metaphysics, it exceeds the meaning of the signs of mathematics and of the natural sciences. The word *number* is precise and definable in mathematics, but when used by the ancient Pythagoreans, for example, *number* not only refers to a mathematical object, but also to the ultimate foundations of the world.

A scientific term is fairly objective when it speaks of a measurable object. A metaphysical term-symbol must be approached differently, however. A metaphysical or religious symbol is understandable only within a history, a tradition, and a community that uses that symbol. The symbol and its context provide its coherence. That context is empirical, for it is made up of history and tradition. What is more, metaphysical symbols can refer even to «Nature». But this is different from the physical measurements of natural science. Metaphysical symbols are, again, mostly determined by the community that uses them to speak of realities beyond what is empirical.

Scientific meaning and metaphysical meaning

In some ways, a scientific hypothesis also is a symbolic picture of how the world might be. But in science, we then try to test and verify that hypothesis. Under these tests, some hypothesis fail and others survive to be tested further. We test the truth of metaphysical and religious formulations differently. They are not subject to empirical testing as we would test scientific principle in a laboratory. Instead, metaphysical and religious formulations seem *true* to us when they offer an intuitive veracity and coherence in the context of the personal values shared by a group of people. Those outside the context will naturally have a very hard time judging the intuitive truth and coherence of a metaphysical viewpoint. As years of interfaith dialogue has shown us, when a Christian states that Jesus is the truth and fullness of God, the Hindu, with another kind of tradition and community, cannot easily comprehend the statement.

Another difference between natural science and metaphysics is that in science, in principle at least, it is said that a hypothesis and theory are never absolute, but only the best approximation so far known in science. In metaphysical perception and language, however, the goal is to arrive at an absolute, a framework that can serve as a permanent vision of reality and the values that flow for it.

Because scientific hypotheses do not try to arrive at the finality known in religion, scientists can hold many metaphysical views. They can be agnostic, atheistic, or theistic in their fundamental beliefs. In other words, a scientist might easily understand the idea that God cannot be tested or proved because God, by definition, is not a mundane object. Religion offers a wider meaning to experience and history, even a timeless meaning.

When metaphysical symbols become articulated, they offer what we call «myths», or grand narratives common to all religions and stories of the origins of the universe, or of how the human predicament came to be. (Myth, in this sense, does not mean an intentional falsehood, but rather a traditional story that tries to meaningfully unite both the facts and mysteries of life). These symbolic narrations are shared by a community and passed on to next generations. The heart of the myths, however, is a set of values that endure even when some of the religious language or narrative by its interaction with newly discovered facts about the world. For example, in Christianity, the stories of the human fall into original sin, and the Resurrection of Jesus, both convey the human struggle with evil and hope for salvation, even though these «myths» can be told in different ways in Christian literature and theology based on our experiences in the world.

To be sure, science has it myths and its narrative as well. One of the most famous is the idea of progress, that science will always and everywhere find true answers and make the right decisions. At the heart of this scientific myth, in other words, is the belief that everything changes, usually in an upward progression. But again, this is a symbolic kind of vision that is hard to test in the laboratory. We seem to accept this myth as true, even though alternative views say that some things may not change, or that progress may be relative and elusive.

As increasing knowledge and communication in our world challenge many of our historical myths — both religious and scientific — the myths that serve us best try to reconcile themselves to the empirical findings of science. Our most helpful myths cannot contradict the scientific data. Religion can accept the knowledge of nature provided by science. In turn, science hopefully can recognize its own necessary openness, and its own myths, and make room for a dialogue with the metaphysical languages that guide our human communities. This is more of a meeting of value system than a battle over the facts of the world. This is not a contest of pure logic, but a discussion of ethics, behavior, and history as well.

Nevertheless, there are fascinating times in our human experience when scientific and metaphysical concepts have overlapped in a very compelling way. As one of the chapters of the book will show, two of those concepts are infinity and what has come to be called the «ontological argument» for the existence of an absolute, that is, for the existence of a supreme being.

3. Historical evolution of mathematics and metaphysical language

About twenty-five centuries ago, humanity began an intellectual ascent that involved the construction and accumulation of the knowledge composing today's systems of mathematics. The summit of this accumulation cannot yet be seen. But we can look back and see the route we have walked.

In the broadest view, this is the story of the development of human rationality. This rationality perhaps has its high watermark in logic and mathematics, but these are not the only realms of human thought that have used the powers of reason. As we saw in the previous chapter, the human ability to ask metaphysical questions has evolved alongside our ability to develop mathematical systems.

I will chart this evolution across four epochs, each with a major characteristic in the development of mathematics. This fourfold scheme begins with the era of primitive mathematics, when its rudiments were first discovered in several cultures, especially in Africa and Asia. Next is the period of the early Greeks, who began to collect a system of mathematics, a system that was expanded upon through the Middle Ages. The third period comes with the scientific revolution of the seventeenth century, when Galileo, among others, declared that mathematics was the language of nature. Finally, in the twentieth century, the language of mathematics is formalized, giving us the tools we use today to analyzing entire mathematical systems and the powers behind our computer revolution.

Sometime in the distant past (up to 5,000 BCE) our ancestors put the first and most basic mathematical capacities into practice. It began with our first perceptions of the plurality of objects and the geometry of shapes. The ability to calculate was the next step in primitive mathematics. A single calculation involves a mechanical procedure; we say it is mechanical because it can now be executed by a machine, such as a computer. Children at school learn calculations in order to find the sum or the product of two numbers.

Before Greek civilization, we have no clear evidence of proofs of mathematical theorems. However, there are earlier accounts that suggest that the idea of the base number was an intuitive knowledge that was put into practice.

The great transition of mathematics from trial and error to a deductive science came with the work of three primary thinkers: Thales of Miletus (d. 548 BCE), Pythagoras (d. 507) and Euclid of Alexandria (d. 265). It is likely that the kinds of new mathematical proofs that we attribute to Thales and Pythagoras, for example, have roots in Babylonian and Egyptian culture. However, it was the Greeks who formally moved from intuition and simple calculation to using first principles — called axioms (and also postulates) — as the starting assumptions needed to prove the validity of mathematical principles.

With the introduction of proofs in mathematics they were also introduced into metaphysics. Metaphysics has attempted to show the rationality of the existence of a supreme being. In all these demonstrations metaphysical intuitions are mixed with logical arguments.

In the seventeenth century, mathematics emerged as a language to talk not just about ideas in the mind, but also about the laws of nature. This began with physics, first looking at objects on earth and then heavenly bodies, and eventually extending to chemistry and biology. Today, the language of mathematics dominates even our social science, that is, our human conduct as studied by sociology and economics.

Modern mathematics and metaphysics

In the centuries before the age of Galileo and Newton, the rational proof of the Greeks was projected into metaphysical systems by the Pythagoreans, the Platonists, and the medieval theologians. In the modern era of mathematics, three new kinds of metaphysical arguments rose to prominence: the principle of sufficient reason, the belief in causal determinism, and the rejection of metaphysics.

In addition to being a mathematician, Leibniz was also a rationalist philosopher in the lineage of René Descartes and others on the European continent. Leibniz is also famous for explicating the idea that nothing occurs in nature that does not have a «sufficient reason» to explain the occurrence. The principle of sufficient reason is equivalent to affirming that God knows all the reasons why everything occurs.

For Leibniz, everything in the world followed a plan of «pre-existing harmony» that was clear in the mind of God. Hence, God had created the best possible world. All aspects of this Leibnizian world, both natural and supernatural, are linked together and can be sufficiently understood by rational methods modeled on mathematics. In talking about the world, Leibniz believed that all men of «good will» could solve problems together by saying, «Let us make the calculations». As advanced and sweeping as Leibniz's vision seemed, he still relied on the basic logic provided by the Greeks in formulating his system of a harmoniously deterministic universe.

The idea of sufficient reason was taken even further by the French scientist Pierre-Simon Laplace (d. 1827), who begins his major work on determinism by citing Leibniz. Laplace, however, seemed even more deterministic than his German predecessor. He said that if everything has a mathematical explanation, everything would be deducible from certain basic mathematical propositions. He was also the successor to Newton in analyzing the solar system. He believed that, based on Newton's laws, all phenomena could be predicted based on the location and momentum of the atoms making up matter. Laplace explained his determinist view by saying that if there were a demon (or an ultimate Mind) that knew the position and momentum of all the atoms in the universe at any given moment, this demon could predict the future in utter detail. Whereas Leibniz attributed the determinism of the world to God's mind, Laplace apparently concluded that modern science no longer needed the «God hypothesis» to explain the workings of the world.

The third consequence of the belief that everything has a mathematical explanation is the complete rejection of metaphysical knowledge. We see this suggested in Laplace, but it was made explicitly by the English philosopher David Hume (d. 1776) in his 1748 essay, *An Enquiry Concerning Human Understanding*. The essay is a spirited attack on all metaphysical thinking. It is also a manifesto for modern empirical science as the only true form of knowledge.

Mathematics Formalized

Up to the twentieth century, mathematics had been gathering every kind of language that was helpful to its work. The first mathematicians put notches on sticks and knots in strings, while centuries later they began to talk about their ideas in shapes they inscribed on papyrus and in their natural languages of Greek, Arabic, Latin, Chinese and Hindi. Eventually, they began to agree on formal signs that could be used in arithmetic, geometry, and algebra, signs such +, –, ×, ÷, which stood for sums, differences, products, and divisions. Letters such as 'a' and 'b', moreover, were used to represent unknowns in mechanical calculations such as $a + b = b + a$.

The final frontier of mathematics, however, was to create a comprehensive formal system that had enough signs and rules to speak of every possible relationship, and to make this system coherent, lacking contradictions. This was the achievement, for the most part, of the twentieth century, although we must say it is not a utopian achievement (since there is no universal agreement on the best formalized system).

The formalization of the language of mathematics has had two important consequences we will review in this chapter. First, such a formalization now allows us to study mathematical language itself by using mathematical methods. Just as mathematics proves theorems about numbers, straight lines, planes, geometrical figures and other mathematical objects, it now can be used to prove the validity of language and meaning of a mathematical system. This is mathematical self-knowledge. It is has allowed us to decisively evaluate the certainty of mathematical propositions.

The second consequence has been truly revolutionary for our practical lives in the modern world. The formalization of mathematical language has produce an automation of mathematical proofs. This automation has connected the human mind with the computer. Once we have formalized mathematical language, it can be put in computer language, and computers, arguably, can calculate and analyze this kind of data with a speed and power that is not possible in ordinary human thinking.

These two consequences take us beyond what mathematics had done in the way of providing representational language for the natural sciences. At the same time, the great debate among professional mathematicians is the validity and coherence of entire mathematical systems. In short, formal mathematics has declared a kind of autonomy from even the empirical sciences. This came about gradually with great milestones in mathematics across the twentieth century.

Truth and certainty

Modern mathematics burst onto the contemporary scene with troubling new questions about the very truth and certainty of older mathematical approaches. In the past, certainty was based first upon intuitions and second upon axioms and proofs. But the consistency of these older certainties began to fall into doubt. Everyone knew that mathematics produced plenty of logical paradoxes, but this fortress was not attacked until the rise of modern mathematics. So the quest began to create a formal system that avoided all paradoxes. Such a system aimed for a complete consistency.

This quest for certainty in mathematics had a strong metaphysical ring, of course. It was an aspiration to find a kind of absolute, unchanging truth. This is the most significant overlap between science and metaphysics in the twentieth century. To find an absolute system, mathematics has tried to reduce all truth to formal signs. This would have a great impact on the more general discussion of how science and religion relate to each other. We will see that the modern idea of truth has come to hinge on the notion of mental consistency. And to the extent that we can agree that consistency is a kind certainty, science and religion can speak to each other of their own internal consistencies and certainties.

Consistency, completeness and decidability

When modern mathematics began to study the nature of its own systems, it was forced to ponder three new questions. Was there consistency in a system? Was a system absolute complete, accounting for all possibilities? And finally, could a system guarantee a decision on any formula, known as the problem of «decidability». The last two of these questions — completeness and decidability — have turned out to be the most problematic for modern mathematical systems. The great minds of the twentieth century have had to conclude that some systems cannot be complete, and some mathematical questions cannot be decided. This suggested that there are limits to truth and certainty. And since mathematics underlies much of science, that uncertainty also extended to the scientific world. As this chapter will show, however, it is finally the enduring existence of consistency that gives our modern age its certainty in math, science, and metaphysics.

In formalized mathematics, we speak of certainties as either univocal or plural. A univocal certainty can be expressed in the same way in all contexts and in all cultures. From the point of view of their formal expression, univocal certainties are the same for everyone regardless of gender, social class, or religion since a formal method can illustrate the case regardless of the cultural context.

On the other hand, formal mathematics also acknowledges the idea of plural certainty. There are the cases where mathematicians cannot reach a common agreement, so a problem cannot ultimately be resolved. The only resolution is to recognize a plurality of systems and how these systems are chosen based simply on the human preferences. Because of this pluralism, mathematicians cannot agree on the philosophical basis of mathematics. In this view, mathematics is much like a house without a foundation. Mathematics floats, so it is more appropriate to speak of mathematical systems as houseboats that shift the positions on the waters of reality.

In the real world, mathematics is made up of univocal and plural arguments and formulas. When there is agreement (univocal) on mathematical certainty, it is usually based on a consistency within a certain formula or system. It is around these certainties that many schools of thought in mathematics gather. However, when there is not agreement, that arises because of the problems of incompleteness and non-decidability of a given formula or system. That formula or system must remain open and cannot be considered an absolute certainty.

In the twentieth century, our recognition that formal mathematics is made up of both univocal and plural realities has had an impact on our view of metaphysics as well.

Metaphysics has traditionally tried to find absolute certainty, conveying that in a symbolic language. The theorems of incompleteness and non-decidability can lead us to think that, in general, it is not possible to make absolute statements of any type.

Nevertheless, beyond the scope of mathematics, every human being needs to ask questions about existence in general and about his own existence in particular. The parallel to this in mathematics is the principle of consistency (technically called principle of non contradiction), which is the widest basis for certainty and truth in mathematics and in life. Mathematics cannot cease to be consistent. Mathematics exists because it is consistent. Therefore, consistency is a logical principle that unites metaphysics and mathematics.

By definition, consistency must exist in mathematics and metaphysics. The attempt to negate (or disprove) a principle, for example, must be carried out with a consistent language since, if this were not so, it would cease to be a logical negation. No matter which way we turn, consistency is required as the most basic logical principle in human thought and communication. Human knowledge of the laws of nature may vary, but the laws of nature — by definition, as laws — cannot cease to be consistent.

As we begin our survey of the formalization of mathematical language in the twentieth century, we will adopt two kinds of overall perspectives. One is that study of the foundations of mathematics has shown that mathematics is plural. The other is that the consistency of thought required in mathematics is where we have found the absolute certainties to reside. With both pluralism and univocal certainty (consistency), we can conclude that mathematics and science remain open to — and often friendly to — the language and ideas of metaphysics.

To understand how the problems of incompleteness and non-decidability arose in the modern history of mathematics, we can survey three stages in the process. This began with the formalization of mathematical language. The second stage was the realization that mathematical systems are incomplete and plural. Third, this process concludes with the findings that, even with the powers of the algorithm, some mathematical calculations cannot be decided. These milestones, and realizations, introduce us to many of the greatest names and personalities in modern mathematics.

First stage: Formal mathematics

One of the first steps that helped the emergence of formal mathematics was the appearance of the non-Euclidian geometries. These showed that geometry did not depend on the traditional intuitions that supported it for so many centuries.

As a byproduct, moreover, this struggle to deal with non-Euclidian geometry forced modern mathematics to deeply question the consistency of all mathematical theories. To attach this problem mathematics need a formal language that could be used between each of the systems to test their consistencies. Hence, the project to formalize all mathematical language was born. That was a language achieved by the German mathematician Gottlob Frege (d. 1925).

On the foundation of those who came before him, the German mathematician David Hilbert (d. 1943) put forward the idea of meta-mathematics, which is the use of mathematical language and principles to analyze mathematical systems themselves as a whole.

Second stage: The incompleteness of arithmetic

Gödel ended up proving that the formal system of arithmetic is incomplete. Nevertheless, the idea of incompleteness does not destroy our assumption that we are being consistent in our thought. It is consistency that led Gödel to his conclusion. He said that if arithmetic is consistent, then it cannot be complete.

The findings of Gödel introduced a crisis to the meta-mathematical program of Hilbert. As a result, all of mathematics had to reassess itself. Was it possible to have a single mathematics, or was the project destined to come up with a plurality of languages and mathematical systems?

Both Hilbert and Gödel based their research and conclusion on the premises of classical logic and mathematics. In that classical tradition, a proposition can be proved true if we can contradict its negation. Furthermore, classical mathematics accepts the existence of infinite sets. These ground rules are what guided Hilbert to seek a universal system, but also what led Gödel to conclude that some component systems of mathematics (such as arithmetic) cannot be complete. Classical mathematics itself led us to this impasse.

So to solve the impasse, other schools of mathematics sprang up. They were based on rejecting some of the basic presumptions of the classic tradition. This explains the origins of the intuitionist/constructivist school of mathematics. This group relies on stricter assumptions about the basis of mathematics. It will accept only those propositions that are directly proved as valid. Depending on the non classical school, the acceptance of sets with infinite elements is also only narrowly accepted, though usually rejected.

Third stage: The formalization of the processes

The idea of a mechanical process is inherent in the uses of informal calculations that were already understood in ancient times. In early logic and algebra, an informal kind of algorithmic processes was employed. What was lacking until the twentieth century was a formal mathematical language to write down the instructions for such a calculation.

Logicians and philosophers had long dreamed of a machine that can calculate all logical deductions. It was in the years before World War II that a computing machine was conceived of to carry out this method based on a set of instructions. Those instructions — the formalization of the algorithmic processes — were the brainchild of two mathematicians at about the same time, Alan Turing (d. 1952) in Britain and Alonzo Church (d. 1995) in the United States.

The mechanism of the Turing machine confirmed that it is not possible to formally prove, in all cases, whether a given program will stop and give an output or not. The Turing machine confirmed mathematical undecidability, strengthening Gödel's theorem of incompleteness.

Pluralism and undecidaility have forced us to recognize the growing complexity of mathematical logic and systems. More and more we are trying to find different solutions to the same problem. The challenge today is to discover criteria to determine the best solutions. The best is always the simplest. But in the application of mathematical formalisms, we are still grappling with a growing complexity of computer programs as we try to process the data of the empirical sciences.

Mathematics, reality, metaphysics

One other way to think of incompleteness and undecidability is that fact that the human mind is creating many of our mathematical systems «ad hoc». Mathematics is not outside the mind, as Platonism and the classical tradition have held. We are next forced to ask whether, if this is true, mathematics can give us a true image of the universe. Or is mathematics a kind of peephole or warped mirror that gives us a limited, or even distorted, view of reality? In our era, the three big questions about mathematics and reality are these: Does math tell us about empirical reality? Can chaotic and probabilistic system in nature ever be intelligible through mathematics? And last, how do we further clarify between the languages of signs and symbols?

Throughout the twentieth century, we have generally adopted a naïve scientific realism that says the knower can simply know the real object. That view, often called positivism or materialism, is now in crisis. In the case of quantum physics, we know that the empirical observations are not totally objective, since our observation interferes with nature in the process. Nevertheless, when these quantum theories are expressed in mathematics, we seem to believe that our observation is objective. In other words, we relay on mathematics to persuade ourselves that we are reading nature objectively.

Before I go into the problematic side of this reliance, we must acknowledge that mathematics has indeed increased our objective grasp of reality. It can formulate law-like principles. Mathematics allows us to make predictions, which are the very basis of scientific research and testing. The same consistency in mathematics produces scientific instruments of exquisite sensitivity and accuracy.

Nevertheless, not only do mathematical systems have limits, but «objective» reality itself is reluctant to allow itself to be controlled. Reality resists our mathematical attempts to enclose it within determinist orders. In chaotic and probabilistic orders of nature, *unpredictability* appears together with predictability. The idea of an «attractor» in a chaotic system illustrates this enigmatic mixture. In a chaotic system, its elements can move to an attractor from different points. The attractor's structure can be deterministic (a normal attractor) or it can be probabilistic (a strange attractor), but in either case, the process reveals order emerging from disorder, something that mathematics is not yet equipped to entirely understand.

In chaotic systems as well, we can observe that a very small variation at some location can substantially alter the trajectory of the entire system, causing it to converge on one kind of attractor or another. Observing this kind of bizarre phenomenon, we naturally wonder about the intelligibility of the world.

Another kind of chaotic system is a «probabilistic order». In this, physical events occur independently of each other. Each event is unpredictable in relation to the others. Nevertheless, there is a probabilistic pattern to chaos that can be captured in mathematical laws such as statistics. Mathematics itself has a kind of randomness and probability as well. By the use of statistics, for example, we can forecast the average distribution of prime numbers even though the position of each prime number seems to be random across the numerical system.

In summary, the chaotic and probabilistic orders have transformed our view of the role of rationality in our empirical study of the world. Like Galileo, we can say that nature is written in a mathematical language. However, unlike Galileo, we can no longer presume to reduce mathematical langue to triangles, circumferences, and other geometric figures. The scientific axioms that we had relied on up until the twentieth century have now lost much of their power of explanation. To know and explain nature today we need to use both signs and symbols.

If mathematics has led the evolution of human rationality across history, then where does our notion of rationality stand today? Whatever that definition will be also will be reflected in our metaphysical outlooks. During the Enlightenment of the seventeenth century, a mathematician such as Leibniz had a deterministic conception of rationality with its doctrine of a «sufficient reason» for everything. Consequently, Leibniz also had a deterministic vision of the metaphysical questions: he viewed God as controlling absolutely every action in the universe.

At the present time, we do not have a deterministic vision of rationality. This will naturally influence our metaphysical questions. Leibniz proposed that, in order to terminate conflicts, when two persons were involved in litigation, they should define the concepts within a formal system of calculation. The parties involved in discussion had simply to sit down and simply make calculations. Leibniz thought that all conflicts could be resolved in this way. Nowadays, we see Leibniz's proposal as naïve. We cannot even perform such calculations within a unique axiomatic system.

The opening of our rational world to risk is not an opening up to irrationality. We are still approaching these metaphysical questions with a consistency that characterizes rational thought. By our rational assessment, we look for the consistency — that is, the lack of contradiction — within each of the plural systems. We can also judge that these systems can co-exist because they do not mutually exclude one another.

To keep rationality in our world, we are required, in the end, to make a metaphysical leap. The formal language of mathematics cannot prove the consistency. Rather, our belief in consistency is a metaphysical presumption. Hence, consistency becomes a rational and scientific value, but also a theological value. Theology must seek to be as consistent as possible with logic, math, and science in a world that is open to incompleteness, undecidabiltiy, risk, and error. This may be the lesson that the history of mathematics offers for the modern dialogue between science and religion.

4. Science, Language, and Religion

Science and its language have had a profound influence on our epoch. Logic continues to be at the heart of our natural languages, and mathematics retains its privileged status as a public language. One simple example of this is how, around the world, we successfully translate a wide array of complex mathematics textbooks into Chinese, Hindi, Malagasy, and other dialects regardless of the language of the original author of the book.

In all of this, our hope has been that logic, mathematics, and science can lift us above our subjective biases to a plateau of objective knowledge and perspective. We seek a public language around the globe. Based on the same rudimentary logic of mathematics, we have developed technology. Thanks to technology, we have cars, aircraft, computers, and refrigerators. We have instruments for the scientific observation and medical treatments. Technology has given our present day culture progressively more power over nature.

But it is perhaps in this area of technology that the wider public is now questioning the powers of our formal and scientific languages. These languages allow us to do many things, but they have also made some aspects of our lives machine-like, automated, and routine. The computer, of course, is the great symbol of where the world may be headed. The computer is a key part of modern industry, and industry is driving such topics of contemporary distress and debate as climatic change, the loss of natural resources, and ecological imbalances in nature. Plato once worried about the modernizing influence of reading (rather than discussion) on young people, and we are seeing a similar concern today in a growing criticism of the language (and power) of science.

Scientific language under fire

Fortunately, human beings have the ability to stand outside of their natural languages and look at them critically. This is what we have achieved in mathematics, for example, with the rise of meta-mathematics: we subject math systems to the very principles they are made of to look for error as well as consistency. It is remarkable to see how a trained monkey, for example, can carry out methodical activities, relating one number to another number and expressing that with sounds. But neither a monkey nor a computer, yet at least, can self-reflect on this type of activity, its meaning, or the language being used.

Throughout the twentieth century, both scientists and philosophers have looked at the historical, psychological, and sociological roots of scientific language and method. For example, it is clear to us now that many kinds of scientific language have developed in history. Each one was based on a different way of knowing, a different epistemology, that is. The language of quantum physics is not reducible to the language of large-scale physics. Quantum physics can only forecast the probably conduct of quantum particles. With the language of macroscopic mechanics, however, we can forecast the movements of bodies with certainty.

One influential work of our time that has pointed out these different kinds of scientific languages, and how they often rival each other, is *The Structure of Scientific Revolutions*, written by the physicist and philosopher Thomas Kuhn (d. 1996). He argues that scientific revolutions take place when anomalies arise in certain conventional system of science and scientific language, and these are rebelled against by a new system of ideas and language. Hence, this is what happened during the Copernican revolution in astronomy, the Darwinian revolution in biology, and the quantum physics revolution in physics. Beginning with the language of observation, all of these revolutions have eventually changed the entire scientific culture of the discussion, often with new languages.

After a long history of science, therefore, we now have a plurality of scientific communities, each with its own chief hypotheses and its own special language. Even though formal logic remains basically at the core of this, the language systems can multiply based on fundamental disagreements about logic or about the problem that a system is trying to resolve. Actually, there is nothing wrong with this critical character in science, for that is what allows us to tests our hypotheses and move closer to valid and useful findings.

Scientific Pluralism

We are left, then, with a plural scientific culture. It has a commonality of method (empirical) and common tools for analysis (logical and mathematics), but it is also divided by its interests and application of these tools. The sciences are diverted to disparate observations of nature. Some look at astronomical objects, others at biological systems, and still others, in the social sciences, at human behaviors. The result is a growing plurality of scientific languages, and due to this, there is often a substantial lack of communication between the different fields.

The most troubling question this has raised for science is the following: How can we trust scientific language to be objective and universal? Some scientists have spoken openly of the limits of objective language when it comes to the subjective experiences that inevitably influence scientific research.

This plurality of languages also extends to mathematics, which is a very human science. There are mathematicians who are committed to classical mathematics with its logical principle of the *tertio excluso*, according to which every formal proposition is true or false. Other mathematicians, on the other hand, hold only that a formal proposition is true or false when we have effectively demonstrated its truth or falseness. Some mathematicians admit the existence of sets with infinite elements. Others only admit the existence of sets which have an arbitrarily large number of elements.

Theology in a scientific culture

The openness in mathematics and science is good news for the language of metaphysics, religion, and faith. Both mathematics and natural science must begin with an assumption, and it is an assumption they chose. That is to say, both disciplines put a certain faith in their assumption and then work outward from that (which we typically call deduction, and is especially characteristic of logic and mathematics). The process is not too different in metaphysics, which includes religion. The medieval scholastics defined theology as *fides quaerens intellectum*, «faith seeking to understand». Similarly, we could that modern science is *perceptio metodica quaerens intellectum*, or «perception and method seeking understanding». Faith (fides) and perception (perception metodica) are parallel experimental ways of obtaining knowledge of human experience, nature, and history.

Science takes in reality through methodical observation. Theology takes in reality through faith. In both cases, the human mind seeks to understand through formulation in a language and the logical structuring of the language. Both scientific theories and theological theories are produced with the common instruments of language and logic. Currently humanity finds many answers in science that our ancestors searched for, and resolved, in religion. This has allowed modern faith to free itself from preoccupations with physical science, a topic that is often not relevant for, or even alien to, what the faith experience is all about.

In the meantime, both the languages of science and metaphysics can be trans-cultural. They can speak to all people despite the barriers of our different natural languages. In science, it is the language of $2 + 2 = 4$ or the laws of gravity. In metaphysics and religion it is the language of absolute reality, such as a transcendent Creator or Principle. The idea of God may be expressed within the context of a culture, but in principle, that culture cannot limit such ideas so that they exclude other human beings who intuit the same higher reality. What is more, in most of our religious traditions, we believe that the Creator reveals this intuitive knowledge to all men and women. For the believer especially, God is seen as actively present in the world. The life of the believer can become a response to a sense that life is a gift, not just a deduction.

Up to the present, both science and religion have shaped our cultures. Our current challenge is to keep a discussion going on between these different kinds of perceptions and languages. Indeed, the history of Christianity (and Judaism and Islam, for that matter) can be viewed as a series of responses to scientific cultures over the ages, from the Hellenistic time through the Enlightenment up to the present age of quantum physics. Today Christianity grapples with these same challenges on the scale of a global scientific-technical culture.

In the past, religions tended to be limited by culture, exposed to only a single culture's internal scientific and philosophical worldview. Now, every religion faces a global scientific-technical culture. This gives an important role to the grand religions such as Judaism, Christianity, Buddhism and Islam, to both understand each other and to adapt themselves as metaphysical options to a scientific age.

A primary way to do this is for the great religions to stay conversant with scientific language, and this helps them to share scientific culture as well. Religions can do this quite safely by recognizing that faith cannot be deduced from empirical knowledge. To quote Augustine, the transcendent cannot be deduced from the immanent («*Si comprehendis non est Deus*»). In effect, the silence of scientific language toward the God question helps purify religious faith. This allows the believer to find harmony between the laws of the world and the presence of a Creator.

Of course, there have been voices in history that have declared that science and religion, because they have opposing perceptions and language, will be in mortal conflict. As the entire book tries to show, the warring ramparts are not as firm as either science or religion, in their more dogmatic eras, once had believed. The world and its natural systems are open, and the transcendent is a logical conclusion we can draw from our consistency of thought. Mathematics and science try to answer «how» things are. Metaphysics and religion try to answer «why» the world is the way it is. We are wise to realize that these questions complement each other, which encourages the delight and curiosity we feel about life. I cannot ask «why» I am in the world if I am not interested in knowing «how» I am in the world.

A central argument of the book is that the language of mathematics holds a privilege status in human affairs. It is a kind of public language that allows us, as best we can, to try to achieve objectivity and certainty. It is more than that as well. Mathematics manifests its knowledge between the extremes of the absolute and nothing. It helps us navigate between the tendency to be subjective or nihilistic, and our tendency to be over-confident and dogmatic. Mathematics shows us that there are certainties — including a kind of logic that makes our natural languages possible — but there also is incompleteness and openness.

For absolute knowledge we must turn to metaphysics and its particular language of symbols in the context of tradition and community. The very fact that this search continues illustrates a kind of sublime, disinterested, and universal consistency in the human mind. This was what prompted some philosophers of the past (in the ontological argument) to try to «prove» God's very existence by logic. We do not need to go that far today — it is enough to acknowledge the consistency of our logical search.

Reconciling the 'Magisteria' of Science and Religion

As a final reflection, I would to like draw on some of the resources of my own tradition of Catholic thought. In that long tradition, we have spoken of the deposit of faith and truth, conveyed by the church in councils, documents and wisdom, as the «the magisterium». It is popularly called the teaching authority of the Church. Not long ago, this topic gained wide attention through the writings of the noted American paleontologist Stephen Jay Gould.

In the 1980s, Gould had been invited to a conference on science in Rome that was organized by the Pontifical Academy of Sciences. He was apparently taken by the idea of a magisterium, or teaching authority. It was this experience in Rome, he later said, that prompted him to propose a model of the relationship between science and religion called the «Non-Overlapping Magisteria», or NOMA. As he wrote:

> «The lack of conflict between science and religion arises from a lack of overlap between their respective domains of professional expertise — science in the empirical constitution

> of the universe, and religion in the search for proper ethical values and the spiritual meaning of our lives. The attainment of wisdom in a full life requires extensive attention to both domains — for a great book tells us that the truth can make us free and that we will live in optimal harmony with our fellows when we learn to do justly, love mercy, and walk humbly… The net of science covers the empirical universe: what it is made of (fact) and why it works this way (theory). The net of religion extends over questions of moral meaning and value. These two magisteria do not overlap, nor do they encompass all inquiry (consider, for starters, the magisterium of art and the meaning of beauty)».

In his desire to avoid conflicts between science and religion, Gould allowed religion its own kind of authority, ethical and moral. But in his desire to give the scientific magisterium a higher ranking in knowledge of the «real» world, Gould was puzzled by the continued effort in Catholic thought to say that science and religion do indeed have «truth» in common. He was particularly puzzled by the 1996 statement of Pope John Paul II, which was titled, «Truth Cannot Contradict Truth».

As an answer to Gould's puzzlement, I would like to offer an alternative to NOMA. I would like to propose the model of a relationship between science and religion called the «Non Symmetrical Magisteria», or NOSYMA. Here, science and religion cannot be separated. Their relationship is complementary, but it is not a symmetrical relationships. Instead, we can say that religious knowledge needs science, while science may do without religion. In effect, this asymmetry is a plus for science by making it autonomous, but it is also a plus for religion by endowing religion with a more comprehensive vision.

In this way, I would argue, the two magisteria cannot be separated. Furthermore, they both use a common language that is characterized by the desire to be logical and consistent. Logic and consistency are what make mathematics the language of science par excellence. The language of theology has also tried to be logically consistent with the language of science.

However, faith cannot close its eyes to mathematics and the empirical sciences. I can separate mathematics from theology, but I cannot separate theology from mathematics. Mathematics and the empirical sciences are independent of religious beliefs, but theological reflection cannot do without mathematics and the empirical sciences.

This asymmetrical model corresponds to how human beings seem to live their lives and how we develop our scientific and religious knowledge. My belief that science and religion complement each other in an asymmetrical way also arises from my own intellectual and spiritual journey in life. I know very good scientists and mathematicians who are believers and others who are non-believers. Since I have chosen faith, I can certainly separate my mathematical work from theology, but I cannot separate theology from what I learn in mathematics.

We also need motivation, hope, and a vision in life, which is something religion offers that science, in proper humility, does not claim to provide. Religion has motivated my scientific search, for example. But I am also aware that science can reject my religion. I have known scientists who have been helped in their work by religion, drawing motivation and inspiration from its wellspring. I've also known others who have renounced religious inspiration in the name of science.

JAVIER LEACH

Universidad Complutense, Madrid
Facultad de Informática
leach@sip.ucm.es

[Artículo aprobado para su publicación en noviembre de 2008]