Miscelánea

Term Structure Estimation: A Review

Autores: *Emma Berenguer*[1]
Universidad Pablo Olvide

Ricardo Gimeno[2]
Banco de España

Juan M. Nave[3]
Universidad de Castilla-La Mancha

Abstract[4]

In macroeconomics and finance, it is extremely useful to have knowledge of the Term Structure of Interest Rates (TSIR) and to be able to interpret the related data. However, independently of its latest particular application, the TSIR is not observable directly in the market and a previous estimation of the yield curve is needed

---

[1] ebercar@upo.e

[2] ricardo.gimeno@bde.es

[3] Juan.Nave@uclm.es

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

125-154

There are two distinct approaches to modelling the term structure of interest rates. The first is essenciallly to measure the term structure using statistical techniques by appliyng interpolation or curve-fitting methods to construct yields. The second approach is based on models, known as *dynamic models*, which make explicit assumptions about the evolution of state variables and asset pricing models using either equilibrium or arbitrage arguments. There is no consensus on any particular methodology and the choice between alternative curve models is, in part, subjective. Nevertheless, the interpolation or curve-fitting methods have showed good properties and are those used nowadays by the vast majority of central banks.

The objective of this article is to present the principal methodologies that have been proposed for the TSIR estimation. A better understanding of the different methodologies and their limitations will provide the researchers with an overview of the problematic of estimation of the yield curve and will enable them to choose the best model according to their objectives.

Keywords: Term Structure of Interest Rates, Estimation Methodologies.

Una panorámica de la estimación de la estructura temporal

Resumen

En macroeconomía y finanzas resulta de gran utilidad conocer e interpretar la Estructura Temporal de Tipos de Interés (ETTI). Sin embargo, independientemente de su aplicación última la ETTI no es directamente observable y es necesario proceder a su estimación.

Existen dos enfoques diferentes para abordar este problema. El primero consiste en obtener la ETTI utilizando técnicas estadísticas de ajuste e interpolación. El segundo enfoque se basa en modelos, conocidos como dinámicos, que utilizan supuestos explícitos sobre la evolución de las variables de estado basándose en argumentos de equilibrio o arbitraje. Al no existir consenso sobre que metodología resulta más apropiada para estimar la ETTI se produce la elección de la misma de forma subjetiva. No obstante, la mayoría de los bancos centrales se decanta por los métodos estadísticos de ajuste de las curvas.

El objetivo de este trabajo es presentar las diferentes metodologías propuestas en la literatura para estimar la ETTI. Un mejor conocimiento de las mismas y de sus limitaciones proporcionará una visión más amplia de la problemática que subyace en la estimación de la ETTI y facilitará la elección de la metodología más adecuada en función de los objetivos que se persigan con la misma.

126-154

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

1. INTRODUCTION

The concept of interest rate belongs to our every-day life. It is generally understood the fact that lending money must be rewarded somehow, so that receiving a given amount of money tomorrow is not equivalent to receiving the same amount today. The interest rates are usually expressed in percentage and referred to a period of time.

The various agents in the real economy are faced with numerous interest rates that are on the basis of which financial and investment decisions are taken. The differences between interest rates in the case of financial assets that share the same characteristics and are generated in the same market should be due, exclusively, to the different term or period associated with each interest rate.This relationship between the interest rate and the term associated with it is known as the *Interest Rate Term Structure* (TSIR).

In macroeconomics and finance, it is extremely useful to have knowledge of the TSIR and to be able to interpret the related data. Following Piazzesi (2010), knowing the term structure is important for, at least, four reasons. The first one is *forecasting*. Since yields on long-maturity bonds are expected values of average future short yields, the current yield curve contains information about the future path of the economy. These forecasts provide a basis for investment decisions. *Monetary policy* is a second reason for studying the yield curve. A model of the yield curve can explain how movements on the short end (controlled, in general, by central banks) translate into long-term yields (important for the aggregate demand). This has as a result understanding both how central bank conducts policy and how the transmission mechanism works . *Debt policy* is a third reason because governments need to decide about the maturity of the new bonds when issuing new debt. A fourth reason is *derivative pricing and hedging*. The price of derivatives as swaps, caps and floors, futures and options on interest rates, is computed from a given model of the yield curve[5]. Hedging strategies also need a yield curve as a basis.

Following its definition, all that is required to obtain the term structure is to take the interest rates of similar assets with different maturities; the problem is that most assets are not homogeneous, neither in their characteristics nor in their maturities. Consequently we are forced to make an estimation of the TSIR.

---

[5] See Duffie *et al.* (2000).

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

127-154

However, independently of the latest particular application of TSIR, there is one concern that is common to the various different approaches, and this is that the TSIR is not directly observable in the market and a previous estimation of the yield curve is needed.

This problem has caused that one of the principal points for the attention of researchers dealing with this topic has been, and still is, to estimate the TSIR from the data provided by the market. There are two distinct approaches to modelling the term structure of interest rates. The first is essenciallly to measure the term structure using statistical techniques by appliyng interpolation or curve-fitting methods to construct yields. The second approach is based on models, known as *dynamic models*, which make explicit assumptions about the evolution of state variables and asset pricing models using either equilibrium or arbitrage arguments. The fact that each approach comprises a considerable number of estimations methods complicates the understanding of the estimation processes.

There is no consensus on any particular methodology and the choice between alternative curve models is, in part, subjective. Nevertheless, the interpolation or curve-fitting methods have showed good properties and are those used nowadays by the vast majority of central banks.

The objective of this article is to to enhance knowledge of the Term Structure of Interest Rates and of the principal methodologies that have been proposed for its correct estimation.Although this is not the first paper dedicated to give a panoramic vision of the yield curve, it is the first including the most relevant estimation methodologies. The pioneer work of Shiller (1990) consolidated and interpreted the literature on the term structure, focusing on definitions, theories and empirical work, but not on modeling the yield curve. Later, Anderson et al. (1996) published a complete handbook dedicated to estimating the term structure. In Spain, there have been several papers related to compiled information on yield curve, (Freixas (1992); Moreno (2000) and Abad y Robles (2003)) but none of them is dedicated to the estimation methodologies.

The rest of this paper is structured as follows: in part 2 some basic concepts related to the term structure of interest rates are introduced; in part 3 the difficulties encountered in estimating this structure are discussed and the principal methodologies developed for making this estimation are described. Lastly, in part 4 the conclusions are presented.

2. THE TERM STRUCTURE OF INTEREST RATES

The term structure of interest rates is usually represented as a continuous curve in time. There are various interest rates - *spot rates, forward rates, return on maturity* (IRR), etc. However, the curve that relates the spot rates to their term is the most

interesting for the various different applications previously cited. This is because the *spot interest rates* reflect only differences based on the different term of the bonds. To the contrary, a curve that relates the *return on maturity* (IRR) with the different terms, is incorporating more information than that strictly due to the term, since the IRR of a bond with a registered coupon is the weighted average (weighted by the cash flows) of the *spot interest rates*. To solve this problem, it is possible to work with the par yield curves, constructed from the yields obtained by the bonds quoted at par; in other words, those bonds whose price is equal to the nominal price, i.e. 100.

Due to their properties, it is common to use the designation of TSIR only (and exclusively) to curve constructed from the *spot rates*.

The spot rates curve reflects the relative situation of the long-term interest rates against the short term interest rates that exists at any particular moment. That relationship gives rise to different shapes of the curve. Thus, the TSIR will be flat when the interest rates are similar for all terms, growing when the short-term interest rates are lower than the long-term rates, and decreasing when the short-term rates are higher than the long-term rates. [6] This last curve is also known as inverted. This is a special situation and occurs when the market is betting on a fall in interest rates in the short and medium term. It is more likely, logically, when the interest rates currently applicable in the market are relatively high. Lastly, it will take the form of an oscillation or hump-back when it presents some rising sections and others falling. Some real examples of the different forms that the interest rates curve can adopt are shown in Annex 1 to this article.

But how are these curves formed and what are the variables on which they depend? There are several theories proposed to explain this.:

The first of these is known as the *Pure Theory of Expectations* and was first developed by Fisher (1896) and subsequently other authors continued the same line[7]. In this theory it is considered that the TSIR is determined only in function of the future interest rates expected by the consensus of the market.

Thus, the *forward* interest rates reflect exactly the interest rates expected for a future moment and, therefore, the long-term interest rates can be conceived as an average of the short-term rates ruling in the future; this means that the expectations regarding these rates influence the form that the TSIR takes at each moment in time.

$$f(t,T) = E_t \left[r(T) \right] \tag{1}$$

According to this theory, it can be stated that the fact that the interest rates curve may be growing indicates that there are expectations of a rise in short-term rates, and if that curve is falling, there are expectations of a decrease in the short-term interest

---

[6] This unstable situation is, however, employed as a working hypothesis in many analyses.

[7] Lutz (1940), Malkiel (1962) and Michaelsen (1963).

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

129-154

rates. A flat structure would indicate a consensus on the stability of the current levels of interest rates.

According to the *Theory of the Preferred Habitat*, due to Modigliani and Sutch (1966) the agents participating in the market have a certain aversion to risk that conditionates their investment strategies, such that they tend to take decisions in accordance with their habitat. However, as their aversion to risk is not total, they take different positions for their investment horizons, provided these positions offer some compensation via risk premiums. This theory offers a fairly complex explanation for the term structure that requires an exhaustive analysis of the supply and demand of securities by terms.

The theory of *Preference for Liquidity*, proposed by Hicks (1939), is a particular case of this approach, in which it is assumed there are risk premiums that are always positive, because the agents are thought to be always adverse to risk and always prefer liquid and short-term investments. Hence, the agents perceive liquidity premiums for supporting the risk inherent in their investments in the long term. Thus, the *forward* interest rates are higher than the short-term rates expected in the future, because there exists a positive risk premium.

$$f(t,T) > E_t\big[r(T)\big] \qquad (2)$$

This theory identifies a rising term structure curve with expectations that future interest rates will be equal to the previous rates, while if a fall in rates is expected, the form of the curve would be decreasing, but only if the expectations manage to nullify the effect of the liquidity. The rising shape of the interest rates curve is also associated with expectations of increases in the spot rates.

In the *Theory of Market Segmentation* it is assumed that there is no systematic relationship between the yields of securities of different terms, and that these yields are determined from the equality between the supply and demand for these securities for each of the terms. The aversion to risk is total and the market agents make the term of their investments coincide with their habitats. Neither the expectations nor the changes that may be considered in the degree of uncertainty influence the evolution of the interest rates. Thus, the term structure is not determined in one single market: rather, it is considered that the structure is segmented in function of the terms. In each segment a process for the determination of prices takes place that is independent, and that takes into account the demand and supply for each particular term. Some of the published studies on this theory are those of Culbertson (1957), Fama (1984) and Mankiw and Summers(1984). Following this theory the analysis that may be made of the TSIR is not relevant, since it does not reflect the expectations of the investors. The *forward* rates are not indicative of any kind of expectation.

One of the most important contributitions is due to Cox *et al.* (1981), who examined this traditional theories about term structure and discussed the implication

of portfolio theory and the valuation of contingent-claims under diffusion processes in this area of research.

In spain, Freixas (1992)[8] and Massot and Nave (2003) have tested the expectation hyphoteses on term structure formation using public debt data, obtainig good results.

Obtaining the TSIR is relevant because it provides information that is very useful for numerous applications. However, it is no trivial matter to obtain this structure, since the fact that it is not directly observable is significant; therefore it must be correctly estimated.

3. ESTIMATION OF THE TERM STRUCTURE OF INTEREST RATES

To be able to obtain for each day the curve that relates the *spot* rates with the different terms with which they are associated, it is necessary to obtain information about the data available in the market, and see if they are sufficient for the construction of a continuous curve.

According to this, the *spot* rates can be obtained directly through the zero-coupon bonds. Therefore, the first task is to identify the markets in which these bonds are traded. Another important aspect is that the interest rates included in the curve should have the same risk, so that the term should be the only difference between them. This last condition is especially important since if this is not the case, we would be incorporating those differences in risk into the securities we were valuing.

In addition to the problem of choosing the source of the original data, another problem arises, since with those data the *spot* rates for a specific term can be obtained, in such a way that we are left with a curve of discrete interest rates, as can be appreciated in figure 1. It is important to obtain continuous curves in time that provide a rate of interest for each term. This makes it necessary to devise a way of filling in the gaps remaining between the interest rates observed, so that a curve for those terms may be obtained (figure 2).

3.1. Available data

From all that has been presented so far, it appears that the most appropriate securities for estimating the TSIR are those of fixed income, specifically those bonds issued by Governments. In fact, from the literature, it is found that these are the bonds that have been employed most assiduously by previous authors for this purpose, although they are not the only ones.

---

[8] Provides an interesting review of the most important papers on this matter.

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

131-154

Figure 1. Euribor for different terms (months)

Source: www.euribor.org

Figure 2. Euribor for different terms (months)

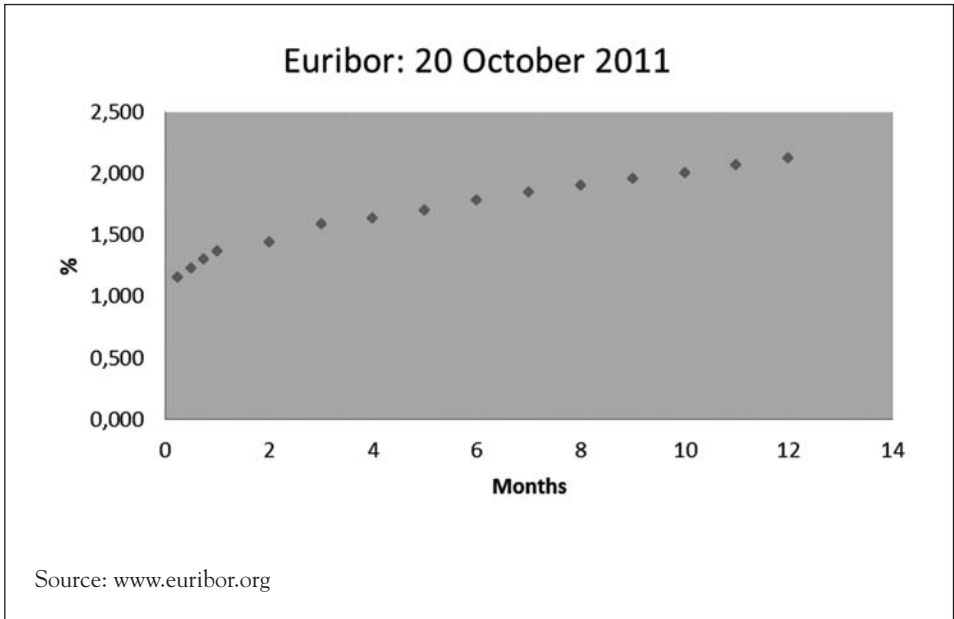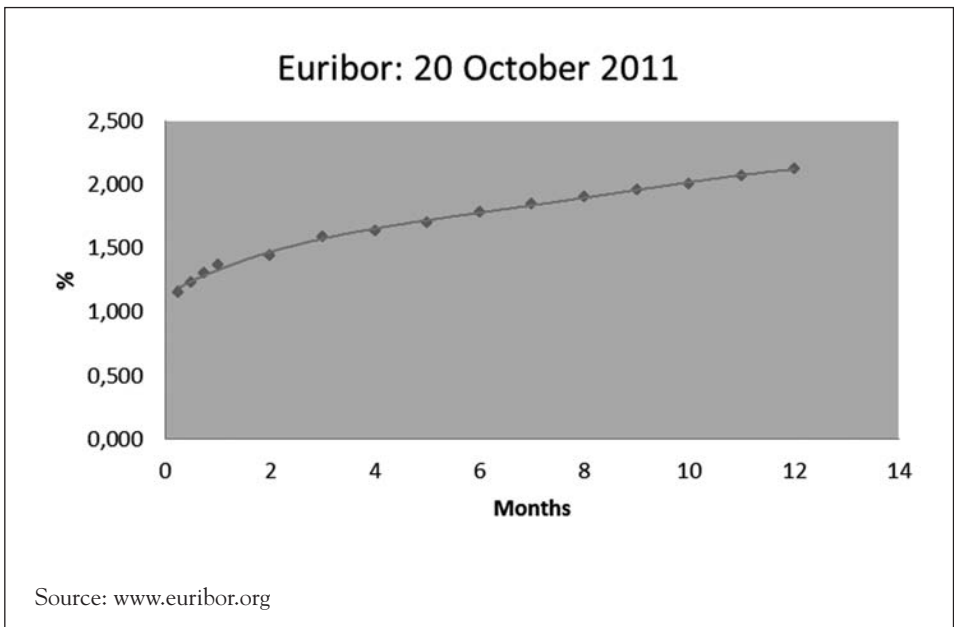Source: www.euribor.org

In the particular case of Spain, there are three markets that have been taken most frequently as the sources for obtaining the data to enable the TSIR to be produced: the interbank market; the secondary market for Public Debt; and the market for Debt *strips*.

3.1.1. The interbank market

This has been used fairly frequently as a source for obtaining the *spot* rates.[9] In particular, *Interest Rate Swaps* (IRS) have been used. Starting with a discount factor for one period a recursive method[10] is employed to obtained the whole curve. To be able to apply this methodology it is necessary to have sufficient references available to allow all the terms to be covered.

However, the curve of *swap* yields obtained this way, presents a serious inconvenience given that these yields incorporate the counterparty risk premiums between the participants of the interbank market in which they are trading. These premiums are not easy to estimate since it would be necessary to know the TSIR free from risk, which is precisely the objective sought. In addition, being a contract between two parties, in some cases the premium would be borne by the variable part of the contract and in others by the fixed part. These securities also originate from an OTC (*over the counter*) market, in which there is no homogeneity in the contracts and in which it is very difficult to undo a position; this removes liquidity from the bonds, and limits the public information on prices.

3.1.2. The market for Public Debt

In Spain, the instruments of Public Debt are Treasury Bills (*Letras de Tesoro*), Bonds and Obligations (*Bonos y Obligaciones del Estado*).

The Treasury Bills are short-term, fixed income securities, are issued by auction, with a minimum amount of each bid of €1,000; bids for larger amounts must be in multiples of that quantity. The Bills are coupon zero (issued to the discount) and the interest they generate will be a term rate (that of the bill)[11]. Being short-term securities, their price variations in the secondary market are usually fairly limited.

Bonds and Obligations are instruments issued for terms in excess of two years[12]. Their characteristics are the same for all designations: the nominal value of each in-

---

[9] See the studies of Leber et al. (2001); Dominguez et al. (2000) and Abad and Novales (2002)

[10] This recursive method is described in Haugen (1997).

[11] Currently the Treasury issues Bills with terms of up to 18 months. Lately, the terms of issue of the Bills have been changing according to the calendar of issues that is fixed each year.

[12] These instruments are issued by means of auction and, the same as with the Bills, their minimum nominal value is €1,000. Currently Bonds are issued at 3 and 5 years, while the Obligations are issued at 10, 15 and 30 years.

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

133-154

strument is €1,000; the interest is paid annually and the amortization is at par value. These securities incorporate explicit yields and are not coupon zero from which the spot interest rates may be extracted directly.

Therefore, the short-term Bills, issued at a discount, can be used directly in the construction of the TSIR, since they are spot interest rates. But the same does not apply with the long-term bonds that pay periodic coupons.

In the market for Public Debt the number of instrument with the coupon zero structure is insufficient to be able to observe the TSIR. In addition, the use of the recursive technique employed with the *swaps* in this market is complicated.

3.1.3. The market for strips

In Spain since 1997, Government Bonds and Obligations can be "segregated". The possibility of segregation allows each bond to be separated into "n" values (called *strips*). Thus, from a 3 year Bond, 4 "strips" might be obtained - one for each payment of annual coupon and a fourth for the principal. This segregation operation transforms an asset of explicit yield (Bond or Obligation) into a series of assets of implicit yield (coupon zero), whose date of maturity and reimbursement value coincide with the coupons and principal of the originating asset. Apart from a more favorable fiscal treatment the rest of the characteristics are identical to the other instruments.

The possibility of segregating these instruments enables the TSIR to be determined directly, although there are several good reasons why its use for the construction of this curve is not advisable.

In practice, the value of a segregated instrument does not coincide with the sum of the *strips* that it generates; this opens up for arbitrage opportunities when the original instrument is reconstructed from its corresponding *strips*. For this reason, the coupon zero curve obtained by means of the *strips* does not correspond with the coupon zero curve that would be desirable and would be obtained from the bonds with coupon.

A second reason is that, since their creation, the volume of these securities in circulation has been increasing, but in no case has covered the expectations created. The proportion of the total public debt accounted for by these securities is very small, in relation to the rest of the instruments.

Analising the data available in the markets, the conclusion must be drawn that the TSIR is not directly observable, because there is not a sufficiently large number of coupon zero types of instrument; therefore, to obtain the TSIR, it must be estimated.

Between the alternatives analyzed, it appears that the best option, due to the properties of the titles, is to start from the daily prices of the Public Debt that are quoted on the secondary market. Then, the next step will be to decide the methodology that will be used to estimate the continuos curve.

3.2. Principal methodologies proposed to estimate the term structure

There are two distinct approaches to modelling the term structure of interest rates. The first is essenciallly to measure the term structure using statistical techniques by appliyng interpolation or curve-fitting methods to construct yields. The second approach is based on models, known as *dynamic models*, which make explicit assumptions about the evolution of state variables and asset pricing models using either equilibrium or arbitrage arguments.

Nevertheless, the statistical models have a number of advantages that make them irreplaceable, these include independence of any preference form and economic assumptions. In this approach yields are function of term to maturity only, without considering a general equilibrium setting or the selection of state variables. A second benefit of these models is that allow simple and parsimoniours functional forms with the advantage of capturing all shapes associated to the yield curve. And finally, these methods are not based on historical data, but instantaneous market quotes, which is very convenient to practitioners. All these reasons, among others, have made the majority of central banks to use these methods instead of those based on equilibrium or arbitraje arguments.

Therefore, we shall pay more attention on the study of methodologies based on statistical techniques.

3.2.1. Statistical techniques

This approach needs to take into account the following:

– What is the function of valuation that relates the price of the bonds with the rate of interest, by means of the promised payments and, perhaps, of other factors.
– What functional form is to be employed to approximate any of the elements that define the term structure? And which of the elements are to be acted upon?
– The econometric method will be used to estimate the parameters of the previously chosen function.

The price of a bond with periodic payment of coupons can be obtained in function of the *spot* rates, as can be seen in the following expression:

$$P = I_1 \cdot e^{-_0 r_1 \cdot 1} + I_2 \cdot e^{-_0 r_2 \cdot 2} + \cdots + (I_n + 100) \cdot e^{-_0 r_n \cdot n} \qquad (3)$$

Where P is the price of the bond, I_s is the coupon paid at moment s, and $_0 r_s$ is the spot rate associated to the period (0,s).

That expression is based on the fundamental theorem of valuation, which implies that if cash flows are known in a market without frictions, the price corresponds to

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

135-154

the current value of these certain cash flows. In this theorem it is held, arbitrage does not exist in the market.

Haley and Schall (1979) analyze the conditions in which the fundamental theorem of valuation is valid, and find that these conditions are those of equilibrium and market efficiency. But in practice, it is known that, when the agents value the various different securities, they are incorporating other factors such as the impositive effect, the interest rate risk, etc., and this effectively takes away validity from the theorem on which the expression (3) is based. So to introduce the possibility that the real price deviates from the theoretical price, this expression must be reformulated to incorporate a perturbation or error term.

This lack of coincidence between the theoretical and the real price finds empirical justification in the study of Carlenton and Cooper (1976), in which the authors demonstrate the impossibility of obtaining a discount function that simultaneously equalizes the contributions of all the bonds with the actual values of their corresponding future cash-flows.

These differences observed between theoretical and real prices find their explanation in the following reasons: i)the quotations of the instruments that are taken as *inputs* in the procedure for obtaining the theoretical prices are the result of an average obtained across the real operations carried out in the markets during one session; this causes differences with respect to the real individual quotations from which this average is obtained.ii)The payment of coupons discretely can also cause deviations between prices, since an approximation will be taken when the accrued coupon is calculated.iii)Other micro-structural aspects of the market in which these instruments are traded.

With all of this, it could be thought that regression techniques could be applied, with the object of regressing the quotations observed in the market, against the matrix of cash-flows.

If the coupon zero rates are taken as the variable, it is observed that the relationship with the price is non-linear, whereas it is linear if one works with the discount function. This does not present any problem, since the previous equation can be rewritten taking into account the discount function, in place of the *spot* rates.

$$P_k = \sum_{s=1}^{n} I \cdot v(s) + 100 \cdot v(n) + \varepsilon_k \qquad (4)$$

To simplify even more the previous equation, the cash-flow that takes place at the point in time *t* will be designated $c(t)$, with $c(n)$ being the sum of the coupon of that last period and the nominal 100. The previous expression would then become:

$$P_k = \sum_{s=1}^{n} c(s) \cdot v(s) + \varepsilon_k \qquad (5)$$

Starting from the previous valuation equation, a system is obtained with as many equations as there are instruments observed in the market. This is a system that would have a solution if the instruments were all to pay coupons on the same dates.

However, this coincidence of dates does not usually occur in the Spanish market for public debt (a situation that does, in fact, apply in the American market for debt, in which the half-yearly cash flows are grouped together on four dates over the course of the year).

Carleton and Cooper (1976) do use this methodology taking advantage of the favorable conditions that the American market offers them for this. Despite this they only manage to obtain data of the discount factors for a term of up to seven years, which seems to be insufficient in a market in which the terms that are traded extend up to 30 years.

The most efficient alternative for countering this problem would be to impose a functional restriction on the discount factors, defining a function that is representative of the data observed. In this approach, what is estimated would be the parameters that characterize the function that tries to approximate the data.

This way of working reduces the number of data to be estimated since, instead of obtaining the curve by means of the estimation of points contained in it, it would be obtained by the estimation of a functional form that defines it from a reduced number of parameters. Thus, substituting the discount factors for the functional form, $G(t)$, the valuation equation would become:

$$P_k = \sum_{s=1}^{n_k} c_{ik}(s) \cdot G(s) + \varepsilon_k \qquad (6)$$

3.2.1.1. Approximation functions

Establishing a functional form would be a good alternative for avoiding problems in the estimation and obtaining a continuous and smooth estimation of the TSIR.

Early work on fitting curves to data from bond market can be found in the doctoral thesis of Guthmann (1929). Subsequently, Durand (1942) made estimations of curves for the French market. Later Cohen *et al.* (1966) and Fisher (1966) investigate yield curve fitting using functional forms.

There are a great variety of functions that can be employed to approximate the TSIR[13]; however, the models proposed allow us to classify those functions in two large groups: *simple functions* and *flexible functions*.

Simple functions

The paper of McCulloch (1971) marked a significant step forward on term structure estimation. In this study the form of the curve can be approximated by means of *polyno-*

---

[13] See Morini (1998).

icade. *Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,* *nº 89 mayo-agosto 2013, ISSN: 1889-7045*

137-154

mial functions; and taking expression (6) as the starting point, it is assumed that $G(s)$ is composed of a linear combination of h functions $g(s)$ linearly independent

$$P_i = \sum_{s=1}^{n_k} c_{ik}(s) \cdot [a_1 \cdot g_1(s) + a_2 \cdot g_2(s) + \dots + a_h \cdot g_h(s)] + \varepsilon_i \tag{7}$$

If the following is defined:

$$W_{ij} = \sum_{j=1}^{h} c_{ik} \cdot g_j(s) \tag{8}$$

From the equality defined in (8), the expression (7) could be represented in the form of a regression equation:

$$P_i = \sum_{j=1}^{h} a_j \cdot W_{ij} + \varepsilon_i \tag{9}$$

where a_j are the parameters to be estimated. From this initial proposal, several different authors have differentiated their proposals on the mode in which they specify the approximation function $G(t)$. McCulloch (1971) himself proposed one of the simpler specifications following the previous methodology:

$$g_j(s) = s^j, \qquad\qquad j = 1, \dots, h. \tag{10}$$

The discount function generated by this set of approximation functions is a polynomial of degree j. However, this function has a single expression for the entire time horizon covered and, unless the number of observations is spread evenly over this period, this gives a better fit in the term for which the most data points are available. Mc Culloch (1971) argues that, although this can be resolved by increasing the degree of the polynomial, this is not the solution, since instability of the parameters estimated is caused. Carlenton *et al.* (1984) also use a simple polynomial form, but rather to describe the instantaneous spot rate of interest[14].

$$_s r_T = \sum_{j=1}^{J} a_{s,j} \cdot t^{j-1} \tag{11}$$

where:

$_s r_T$ is the instantaneous spot rate for the term *T-s*
t is the time *T-s*
J is the number of polynomials
a_{s-j} is the coefficient of the *j*-th polynomial at the time *s*.

---

[14] Due to these problems he proposes using flexible functions.

Thus the discount function would be expressed in the following way:

$$v_s(t) = e^{-\sum\limits_{j=1}^{J} a_{s,j} \cdot t^j} \tag{12}$$

and the regression equation:

$$P_{i,s} = \left[\sum_{t=1}^{T} c_i \cdot (s+t,s) \right] \cdot e^{-\sum\limits_{j=1}^{J} a_{s,j} \cdot t^j} + \varepsilon_{i,s} \tag{13}$$

To make this estimation, Carlenton *et al.* (1984) use non-linear, least-squares regression techniques. With this regression their object is to determine what degree of the polynomial is the most appropriate; and for this they compare the results using different degrees, between one and five. The conclusion they reach is that the most appropriate degrees for estimating the term structure, within the time horizon of the sample, are three and four.

On analyzing the results obtained with this estimation procedure, Carlenton *et al.* (1984) conclude that the results are by no means as acceptable as one would expect. For this reason they repeat the regressions without including the data relating to bonds with periodic payment of coupons, and they use the method of maximum likelihood, working now with sixth degree polynomials. By this means they obtain better results, but at the cost of complicating the methodology employed.

Another proposal for approximation by means of a simple function is that of Nelson and Siegel (1987); their proposal is based on the concept of *forward* rate that is described in the theory of expectations. Nelson and Siegel (1987) assume that the *forward* rates implicit for any term tend asymptotically towards a particular level; in other words, in the long term these are almost identical.

If the instantaneous forward rate of interest, f(t), is the solution to a second-order differential equation whose characteristic polynomial has different real roots, the following is given:

$$f(t) = \beta_0 + \beta_1 \cdot e^{\left(\frac{-t}{\tau_1}\right)} + \beta_2 \cdot e^{\left(\frac{-t}{\tau_2}\right)}; \tag{14}$$

where β_0, β_1 and β_2 are determined by the initial conditions and τ_1 and τ_2 are time constants associated with the equation.

This equation generates a family of *forward* curves that includes all the possible regular forms of the TSIR. Nelson and Siegel (1987) observe that modifications in the value of τ_1 and τ_2 did not improve the fit; therefore they eliminate one of the

icade. *Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales*, nº 89 mayo-agosto 2013, ISSN: 1889-7045

139-154

constants ($\tau_1 = \tau_2 = \tau$) and proceed to define the instantaneous forward rates using the next equation:

$$f(t) = \beta_0 + \beta_1 \cdot e^{\left(\frac{-t}{\tau}\right)} + \beta_2 \cdot \frac{t}{\tau} \cdot e^{\left(\frac{-t}{\tau}\right)}; \qquad (15)$$

In which τ, β_0, β_1 and β_2 are the parameters to be estimated.

Nelson and Siegel (1987) propose the testing of the model employing ordinary least squares, but it is also possible to use techniques of non-linear estimation (non-linear minimum squares or maximum likelihood)[15].

Svensson (1994) modifies the original work of Nelson and Siegel (1987). For this, he recovers the second term, which Nelson and Siegel (1987) had disposed of, and he also adds to it another parameter (τ_2). Thus, the equation that defines the instantaneous *forward* rate is the following:

$$f(t) = \beta_0 + \beta_1 \cdot e^{-\frac{t}{\tau_1}} + \beta_2 \cdot \frac{t}{\tau_1} e^{-\frac{t}{\tau_1}} + \beta_3 \cdot \frac{t}{\tau_2} e^{-\frac{t}{\tau_2}} \qquad (16)$$

The inclusion of this additional term allows the existence of more than one optimum, and so the curve is able to present a maximum and a minimum simultaneously.

Svensson (1994) estimates all the parameters by maximum likelihood, although he considers that non-linear least squares or the method of the generalized moments could also be employed.

As Gimeno and Nave (2009) state, the principal disadvantage of this method is the correlation between the estimators of the parameters, already marked in the model of Nelson and Siegel (1987), which is further strengthened, in this case.

Diament (1993) obtain the yield curve directly from the set of prices of the most liquid instruments of the market (*on the run*), that are the last issued for each maturity. As there are only nine of these directly observable instruments, this author resorts to a method of interpolation to construct a continuous curve. A semi-empirical approximation is made to arrive at an adequate interpolation to be able thus to construct its smooth and continuous fit of the discrete term structure.

The equation to obtain the term structure is:

$$r(t) = \frac{C_1 \cdot (t/C_3)^{C_4} + C_2}{(t/C_3)^{C_4} + 1} \qquad (17)$$

Where C_1, C_2, C_3 and C_4 are the 4 parameters to be estimated.

---

[15] For this to be possible, the value of the parameter that affects the linearity of the model, , has to be established externally. Or alternatively, to have available bonds without coupons or to work with IRRs.

Diament (1993) bases his method on a model of four parameters that correctly fits the rising and falling trends and even the flat curves, but needs modifications in the event of any "humpback" being observed; that problem is resolved by adding two more parameters.

The model proposed by Diament (1993) presents some disadvantages, perhaps the most important of these being that it is essential, previously, to observe the form of the curve in order to decide if it is monotone or not and, in consequence, to apply the functional form of 4 or 6 parameters, according to the case[16].

Mansi and Phillips (2001) propose a model based on an exponential function that depends on the estimation of four parameters. It represents an improvement over the model proposed by Diament (1993), since it does not require previous knowledge of the shape of the curve to estimate the term structure. The functional form is the following:

$$r(t) = D_1 + D_2 \cdot e^{D_4 \cdot t} + D_3 \cdot e^{2 \cdot D_4 \cdot t} \qquad (18)$$

Table 1. Studies based on simple functions

Principal studies that estimate the TSIR by means of simple functions			
Author/es	Year	Methodology	Function approximated
McCulloch, J.H	1971	Polinomial functions	Discount function
Carlenton, W.T; Chambers, D.R y Waldman, D.W	1984	Polinomial functions	Spot rate curve
Nelson, C.R y Siegel, A.F	1987	Functional form	Implied forward rate
Diament, P	1993	Exponential function	Par yield curve
Svensson, L.E.O	1994	Functional form (extended version of Nelson & Siegel)	Implied forward rate
Mansi, S y Phillips, J	2001	Exponential function improving Diament's	Par yield curve

3.2.1.2. Flexible functions

Even though simple functions are used widely for the estimation of the TSIR, they are not free from disadvantages originated, mostly, from the instability of the parameters estimated. The excessive degree used in the polynomials when approximating the whole of the interval reduces the smoothness of the curve. Because of this, a way is sought to reduce the degrees of the polynomial without losing accuracy. This is achieved by approximating discrete sections of the interval, rather than the whole of it.[17]

---

[16] It is not that the case of six parameters cannot always be used; rather, if it is not used correctly, it generates problems of convergence.

[17] See Shea (1984)

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

141-154

A *spline* function is a continuous function in pieces or sections (subintervals).It is a function defined differently for each subinterval. The points where the sections are joined together are known as knots, in which the functions should have the same value to ensure that the single function is continually differentiable. The following condition must be met:

$$S(x) = f_i(x), \text{ for } x_i \leq x \leq x_{i+1} \rightarrow \qquad S(x) = \begin{cases} s_0(x) = f_0(x), \text{ for } x_0 \leq x \leq x_1 \\ \vdots \\ s_{n-1}(x) = f_{n-1}(x), \text{ for } x_{n-1} \leq x \leq x_n \end{cases} \qquad (19)$$

To define a *spline* formally, a partition is made in the interval under study [a,b]. A= {a= x_0<x_1<…<x_n=b}

A *spline* ($S(x)$) of degree r defined over A, is a real function, $S(x)$:[a,b]→R which complies with the following properties:

- $S(x) \in C^{r-1}[a,b]$; that is, $S(x)$ is derivable with continuity up to the order r-1 in the interval [a,b].
- $S(x)$ is, for each subinterval[x_i-x_{i+1}], a function of, in sum, degree r.

There are two ways of constructing a *spline* function: the first is from the conditions of continuity that this must meet in the knots; and the second is making use of a base. In this latter case, the *spline* is expressed as a linear combination of the functions that comprise the base[18].

The *Basic splines*(B-splines) are very flexible functions; they are noted as B_i^r with r being the degree of the *spline* and i the subinterval to which it refers. They are simple to obtain, as they are based on a relationship of recurrence in function of the *B-spline* of the immediately lower order. The *B-spline* functions were initially applied for approximating the discount function by means of a linear regression. In accordance with Powell (1981), the *B-spline* function is defined as:

$$g_s^p(t) = \sum_{l=s}^{s+p+1} \left[\prod_{j-s, j \neq i}^{s+p+1} \frac{1}{(T_j - T_i)} \right] [máx(t - T_{j,} 0)]^p \qquad (20)$$

where $g_s^p(t)$ is the s-th (s=1, 2, …, p+m) *B-spline* function of order p.

Its value is different from zero only if t belongs to the interval [T_s , T_{s+p+1}], and it takes the zero value in the rest of the cases. The number of subperiods between t=0 and the longest maturity of the bonds of the sample is m. For this reason, p+m *B-spline* functions are required. The end-points of each time interval [T_s, T_{s+1}] are known as knots. The number of knots required will be 2 p+m+1.

---

[18] De Boor (1978) will be found very illustrative on this point.

The number of knots of the interval is one of the most important factors to take into account, since an increased number of knots leads to greater flexibility of the resulting curve, but too many causes a loss of smoothness; however, with too few knots the curve does not fit correctly. It can be observed that the choice of the number of knots in the construction of splines implies an incompatibility between two of the most important characteristics in the estimation of the TSIR: fit and smoothness. For this reason, the determination of this number must be resolved by using *ad-hoc* methods. In the studies of Steeley (1991), Lin (2002) and Ramponi (2003) this problem of number and positioning of knots is given particular attention.

In continuation, several of the most interesting studies on TSIR estimation based on splines are discussed.

McCulloch (1971) was the first to use *spline* functions in the fit of the TSIR; specifically this author used second degree polynomial *splines* to fit the discount function. After failing to obtain a sufficiently smooth curve, in a later study, McCulloch (1975) raised the degree of the *spline* by employing cubic polynomial *spline* functions. However, this also failed to resolve the problem completely, since the employment of cubic splines gave the curve of forward rates more smoothness at the cost of producing, on occasions, negative forward rates.

Another methodology, used in parallel with the introduction of splines, has been the use of the polynomial approximations of Berstein. This has been used in order to avoid the problem demonstrated in McCulloch's work (1975) in relation to the use of polynomial splines in the fitting of the *discount function* and, in particular, to its rigid behavior[19]. In this line, Schaefer (1981) uses the methodology developed in Schaefer (1979) in which the functions used are derivations of Berstein-type polynomial functions, whose positive combinations approximate decreasing monotone functions with sufficient accuracy, and give rise to considerably better approximations of the curves of *forward* rates than those from which they are derived. However, as noted by Shea (1984), they do not guarantee the stability of the *forward* rates curve: very sharp changes can occur in their short-term section.

Another of the more representative proposals of the use of splines is that of Vasicek and Fong (1982); this is based on the fact that the *discount function* usually adopts expressions of the exponential type. These authors approach the estimation of this function by using exponential spline functions. In particular, they carry out a transformation of the argument of the *discount function* $v.(t)$, that is approximately exponential with respect to time, in an approximately potential function, $G(x)$, with respect to the new variable.

---

[19] De Boor (1978) demonstrates that the approximation of a function in an interval $[a, b)]$ using *B-splines* is equivalent to a Berstein polynomial approximation in which: $n=K$ and $[a, b)] = [0, 1]$

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

143-154

They propose the following transformation:

$$t = -\alpha^{-1}\ln(1-x), \qquad 0 \le x \le 1, \qquad (21)$$

where α is a constant[20], and hence $G(x)$ is then defined as:

$$G(x) \equiv v(t) = v(-\alpha^{-1}\ln(1-x)). \qquad (22)$$

The model is defined initially from the discount function; it also incorporates an adjustment to take account of the impositive effect (Q) and the possibility of early cancellation (W)[21]:

$$P_k = 100 \cdot v(t_n) + \sum_{j=1}^{n} I_k \cdot v(t_j) - Q_k - W_k + \varepsilon_k \qquad (23)$$

This equation can be rewritten using the transformed function $G(x)$ expressed as a linear combination of a base of polynomial *splines* whose parameters are estimated by least squares. The parameters thus estimated depend on the value of α, which is later optimizedusing numerical procedures. The study of Vasicek and Fong (1982), despite its important contribution to the modeling of the TSIR, has been severely criticized[22] due, principally, to the lack of empirical analyses that would corroborate its statements, and due to the lack of transparency in some aspects of the methodology. A methodology similar to this is proposed by the Bank of England (Mastronikola 1991); in this approach a cubic polynomial function is defined for each subinterval of the function of yields at par, introducing the restrictions of equality of the value of the contiguous functions in the knots and of the first two derivations, in order to produce smoothness in the fitted curve.

Shea (1984) demonstrates in his work several disadvantages of the *spline* functions used by McCulloch (1971) and by Vasicek and Fong (1982) in the smooth estimation of the TSIR. One of these is the multicollinearity existing in the regression matrix. The solution he proposes to avoid them is the use of the *B-spline* bases defined previously. The proposal of Shea (1984) is shared by Steeley (1991), who, in addition to recommending the use of the B-splines, reproduces all the steps necessary for their construction by means of a very illustrative example.

Fisher et al. (1995) approximate the curve of forward rates by means of splines that they designate as "smooth", using a cubic B-spline as base. That smoothness is obtained with the incorporation of a penalty for variability, materialized in a single parameter. An increase in the penalty reduces the number of parameters of the *spline*, in such a way

---

[20] It represents the value of the *forward* interest rates when t tends to infinity. See Contreras *et al.* (1996)

[21] Specifically Q_i is the decrease in the price of the instrument k due to the impositive effect and W_k represents the reduction in the price of the instrument k due to the early amortization.

[22] See Shea (1984) and Deacon and Derry (1994).

that it is the data that determine the number of appropriate parameters of the *spline*. The mathematical specification of smoothness that they use is that defined earlier by Adams and Van Deventer (1994) except for the appearance of the constant λ.

$$Z = \lambda \cdot \int_0^T [f''(s)]^2 \cdot ds \qquad (24)$$

One disadvantage to take into account in the methods of Adams and Van Deventer (1994) and of Fisher *et al.* (1995) is that the penalty reduces the oscillatory behavior but also reduces the accuracy of the fit. Bliss (1996) studies the model presented by Fisher et al. (1995) and considers that better results could be obtained if λ depended on the maturity; in particular if λ (t) were growing for t.

Waggoner (1997) works with a parameter of penalty that is variable between maturities. This author specifies a small penalty in the short-term and a greater value in the long-term, allowing there to be sufficient flexibility in the short section, but preventing oscillatory movements from occurring in the long section. Later Anderson and Sleath (2001) adapt the proposal of Waggoner (1997) to the Public Debt market of the United Kingdom.[23] This method is known as the Variable Roughness Penalty (VRP) and is the method currently used by the Bank of England.

Given a especification of the functional form of the yield curve, the estimation procedure consists on minimizing the sum of squared differences between the observed and the fitted yields.

Table 2. Studies based on flexible functions

Principal studies that estimate the TSIR by means of flexible functions			
Author/es	Year	Methodology	Function approximated
McCulloch, J.H	1971	Quadratic spline	Discount function
McCulloch, J.H	1975	Cubic spline	Discount function
Vasicek, O.A y Fong, H.G	1977	Exponential splines	Discount function
Schaefer , S.M	1981	Berstein polynomial	Discount function
Mastronikola,	1991	Cubic spline	Par Yield
Steely	1991	B-splines	Discount function
Fisher, M; Nychka, D y Zervos,D	1995	Cubic B-spline Penalty for variability (λ)	Implied forward rate
Waggoner, D.F	1997	Penalty for variability dependent on the maturity λ (t)	Implied forward rate
Anderson N y Sleath, J	2001	Variable Roughness Penalty (VRP)	Implied forward rate

---

[23] Given the different characteristics of the American Public Debt market, they consider λ(t) a continuous function of three parameters.

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

145-154

3.2.2. Dynamic Models

In this models, prices are driven by random processes which by their uncertain nature are risky. This is formalized by defining state variables characterizing the state of the economy (relevant to the determination of the term structure) which are driven by these random processes and related to the prices of the bonds. This kind of term structure models are well-known as *Affine Term Structure Models (ATSM)*[24].

Since the seminal models of Vasicek (1977) and Cox, Ingersoll and Ross (1985), known as CIR (1985) dynamic models have been widely used to price fixed-income assets.

Vasicek (1977) model consist of a single state variable which is conveniently chosen to be the short rate. This is assumed to evolve as:

$$dr_t = k \cdot (\varphi - r_t) + \sigma \cdot dB_t \tag{25}$$

where φ is the asymptotic mean of the short rate, k is the rate of mean reversion and s determines the local volatility of the process for a given value of r_t.

CIR (1985) develop an intertemporal general equilibrium model of asset prices. They apply this framework to construct one-and two-factor models of the tern structure. In the one-factor model they define a state variable for the short rate which evolves according to:

$$dr_t = k \cdot (\varphi - r_t) + \sigma \cdot \sqrt{r_t} dB_t \tag{26}$$

Despite their different equilibrium assumptions, this model differ from Vasicek's in the squared-rootterm in the local volatility.

These one-factor models have the problem of restricting the dynamics of term structure and, in consequence fit the term structure quite poorly. To solve this problem some models increased the number of state variables driving the term structure, (Duffie and Kan (1996); Dai and Singleton (2000)) while others try to make more flexible the short interest process (Ho and Lee (1986); Hull and White (1990); Heath, Jarrow and Morton (1992)).

Following the distinc approaches a aconsiderable number of *dynamic models* were proposed. We recommend for a complete literature revision of these models to consult Piazzesi (2010).

Navarro and Nave (1997) proposed a two-factor model based on the methodology suggested by Elton *et al.* (1990) in order to select the best spot interest rate to be used for describing the Spanish TSIR according to the composition of the major Spanish mutual funds specialized in Public Debt assets.

---

[24] A complete revision of these models and their properties can be found in Piazzesi (2010).

146-154

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

More recently,the seminal paper of Diebold y Li (2006), focused on forecasting the yield curve, use neither the no-arbitrage approach not the equilibrium approach. Instead, use variations on Nelson-Siegel exponential component framework to model the entire yield curve, period-by-period, as a three dimensional parameter evolving dynamically.

Based on the methodology of Diebold and Li (2006) the paper of Gimeno and Marques (2012) defines, for the Spanish Market, related dynamic models in which the latent factors are replaced by the parameters of the model of Nelson and Siegel (1987),adding the lack of arbitrage opportunities and risk aversion.

In the Spanish Market, De Andrés and Terceño (2003)[25] think that modeling the behaviour of interest rates in the long term by means of a stochastic model is not very realistic and thus, proposed to make financial analyses in the long term using fuzzy estimates of the discount rates. In their paper, they attempt to provide a solution to this question with a fuzzy regression method for adjusting the temporal structure of interest rates (TSIR).

4. CONCLUSIONS

The main objective of this article is to enhance knowledge of the Term Structure of Interest Rates and of the principal methodologies that have been proposed for its correct estimation.

The range of studies and methodologies proposed is very wide. All the studies discussed seek the same objective: to estimate the term structure of interest rates in such a way that the curve obtained is smooth, flexible and stable.However, there is no consensus on one methodology in particular that is generally accepted as superior to the rest. The fundamental aim is to obtain an estimation of the TSIR which is as reliable as possible; however, the degree of accuracy required will be determined by its ultimate application.

There are two distinct approaches to modelling the term structure of interest rates. The first is essenciallly to measure the term structure using statistical techniques by appliyng interpolation or curve-fitting methods to construct yields. The second approach is based on models, known as *dynamic models*, which make explicit assumptions about the evolution of state variables and asset pricing models using either equilibrium or arbitrage arguments.

Given this lack of consensus, there are many studies that compare a variety of methods for testing empirically which method best approximates the TSIR. On this point, the number of studies is so large and their characteristics so heterogeneous

---

[25] The previous paper of De Andrés (2000) had applied the model of McCulloch (1971) using fuzzy regressions.

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

147-154

that it is not always possible to obtain clear conclusions. However, with reference to the comparison between splines and parametric models, there are many studies that obtain better results with these latter[26].

Furthermore, between the different methodologies that have been presented here, the methods based on simple or parametric functional forms present a series of advantages over the *splines* that make them very attractive for estimating the TSIR. Among those advantages, the following can be emphasized:

– The parameters of the models have economic meaning, which facilitates interpretation of the results obtained.
– They do not have the risk of 'over-parameterization' that is a characteristic of the models based on *splines*. This problem, which arises when working on different sections of the curve, can result in negative forward interest rates being obtained, which could lead to erroneous analyses.
– To avoid the previous problem, the majority of the central banks choose to use models of the parametric type in their estimations (Annex 2).
– It is simpler to work with parametric methods than with *splines* because, among other reasons, all the difficulties with the positioning of the knots are eliminated. Most of the methods based on *splines* have shown great sensitivity with respect to the selection and positioning of the knots.

From this point, some further empirical research is needed because the amount of papers dedicated to compare parametric models in Spain is very limited and outdated.

REFERENCES

ABAD, P. and NOVALES, A. (2002), *Risk premia in the term structure of swaps in pesetas*, Serie ICAE Documentos de trabajo de la Universidad Complutense de Madrid.

ABAD, P. and ROBLES, M.D. (2003), "Estructura temporal de los tipos de interés: teoría y evidencia empírica", *Revista Asturiana de Economía*, 27.

ADAMS, K. and VAN DEVENTER, R. (1994), "Fitting yields curves and forward rate curves with maximum smoothness", *The Journal of Fixed Income*, Junio, pp. 52-62.

ANDERSON, N., BREEDON, F., DEACON, M., DERRY, A. and MURPHY, G. (1996), *Estimating and interpreting the yield curve*, New York: Wiley.

---

[26] See Bliss (1996) and Ioannides (2003). Specifically for the Spanish Market see Nuñez (1995) and Berenguer (2009).

ANDERSON, N. and SLEATH, J. (2001), *New estimates of the U.K real and nominal yield curves*, Bank of England Working Paper Series .

BANK FOR INTERNATIONAL SETTLEMENTS (BIS) (2005), *Zero-coupon yield curves: technical documentation*. BIS papers 25.

BERENGUER, E. (2009), *La Estimación de la Estructura Temporal de los tipos de interés: Metodología y Aplicaciones.* Unpublished PhD dissertation. Universidad Nacional de Educación a Distancia.

BLISS, R. (1996), "Testing term structure estimation methods", *Working Paper* No. 96-12, Federal Reserve Bank of Atlanta.

CARLENTON, W. and COOPER, I. (1976). "Estimation and uses of the Term Structure of Interest rates", *The Journal of Finance*, 31.

CARLENTON, W., CHAMBERS, D. and WALDMAN, D. (1984), "A new approach to Estimation of the Term Structure of Interest rates", *Journal of Finance and Quantitative Analysis*, 1 ,pp. 233-252.

COHEN, K., KRAMER, R.L. and WAUGH, W.H. (1966), "Regression yield curves for U.S government securities", *Management Science*, 13. December.

COX, J., INGERSOLL, J. and ROSS, S. (1985), "A theory of the term structure of interest rates." *Econometrica*, 53,March, pp. 385-407.

_ (1981), "A re-examination of traditional hyphothesis about the term structure of interest rates". *Journal of Finance*, 36, pp. 769-799.

CULBERTSON, J. (1957), "The term structure of interest rate", *Quaterly Journal of Economics*, 71, pp. 485-517.

DAI, Q. and SINGLETON, K. (2000), " Specification analysis of affine term structure models", *Journal of Finance*, 55,pp. 1943-1978.

DE ANDRÉS, J. (2000) *Estimación de la estructura temporal de los tipos de interés mediante números borrosos*, Unpublished PhD Dissertation. Universidad Rovira i Virgili.

DE ANDRÉS, J. and TERCEÑO, A. (2003). "Estimating a term structure of interest rates for fuzzy financial pricing by using fuzzy regression methods", *Fuzzy Sets and Systems*, 139(2), pp. 313-331.

DE BOOR, C. (1978), *A practical guide to splines.* New York: Springer-Verlag.

DEACON, M. and DERRY, A. (1994), "Estimating the Term Structure of interest Rates", *Bank of England Working Paper Serie, nº.24.*

DIAMENT, P. (1993), "Semi-empirical smooth fit to the Treasury Yield Curve", *Journal of Fixed Income*, Junio, pp. 55-70.

DIEBOLD, F.X and LI, C. (2006), "Forecasting the term structure of government bond yields", *Journal of econometrics*, nº130, pp. 337-364.

DOMINGUEZ, E. and NOVALES, A. (2000), "Testing the Expectations Hyphotesis in Eurodeposits", *Journal of International Money and Finance*, 19, pp.713-736.

DUFFIE, D., JUN, P. and SINGLETON, K. (2000), "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, 68, pp.1343-1376.
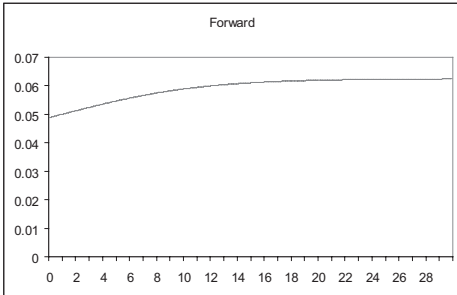
icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

149-154

DUFFIE, D. and KAN, R. (1996), "A yield-factor model of interest rates", *Mathematical Finance*, 25, pp. 379- 406.

DURAND, D. (1942), "Basic yield of Corporate Bonds, 1900-1942", *Technical Paper*, Vol.3 NBER.

FAMA, E. (1984), "The information in the term structure", The Journal of Financial Economics, pp. 509-528.

FISHER, I. (1896), "Appreciation and interest", *Publications of the American Economic Association*, pp. 23-29.

FISHER, M., NYCHKA, D. and ZERVOS, D. (1995), "Fitting the term structure of interest rates with smoothing splines", *Finance and Economics Discussion Series 95-1*, Federal Reserve Board. Washington D.C.

FREIXAS, X. (1992), Estructura temporal de tipos de interés: hipótesis teóricas y resultados empíricos, *Investigaciones Económicas,* 16, pp. 187-203.

GIMENO, R. and MARQUÉS, J.M. (2012), "A market based approach to inflation expectations, risk premia and real interest rates", *Spanish Review of Financial Economics*, vol. 10, pp. 18-29.

GIMENO, R. and NAVE, J. (2009), "Genetic algorithm estimation of interest rate term structure", *Computational Statistics and Data Analysis*, 53, pp. 2236-2250.

GUTHMANN, H. (1929), *The relation of the maturity factor to yield differences among investment securities.*Unpublished PhD Dissertation: University of Chicago.

HALEY, C. and SCHALL, L. (1979), *The Theory of Financial Decisions.*MacGraw Hill.

HAUGEN, R.A. (1997), *Modern Investment Theory*. Prentice-Hall, Inc.

HEATH, D., JARROW and MORTON, A. (1992), "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation", *Econometrica*, 60, pp. 77-105.

HICKS, J. (1939), *Value and Capital.* London.: Oxford University Press.

HO, T.S. and LEE, S. (1986), "Term structure movements and pricing interest rate contingent claims", *Journal of Finance* , 41, pp. 1011-1029.

HULL, J. and WHITE, A. (1990), "Pricing interest-rate derivatives securities", *Review of Financial Studies,* 3, pp. 573-592.

IOANNIDES, M. (2003), "A comparison of yield curve estimation techniques using UK data", *Journal of Banking and Finance*, 27, pp. 1-26.

LEBER, M., FERNÁNDEZ-NAVAS, J. and SOLER, J. (2001), "Valoración de bonos en el mercado de deuda del Estado con modelos estáticos y dinámicos de la ETTI", *Revista Española de Financiación y Contabilidad XXX..*

LIN, B. (2002), "Fitting term structure of interest rates using B-splines: the case of Taiwanese government bond", *Applied Financial Economics*, 12, pp. 57-75.

LUTZ, F. (1940), "The structure of interest rates", *Quaterly Journal of Economics*, pp. 36-63.

MALKIEL, B. (1962), "Expectations, bond prices and the term structure of interest rates", *Quaterly Journal of Economics*, 76, pp. 197-218.

MANKIW, N. and SUMMERS, L. (1984), "Do long-term interest rates overreact to short term interest rates", *Brooking Paper in Economic Activity*, pp. 223-247.

MANSI, S. and PHILLIPS, J. (2001), "Modeling the term structure from On-the-Run Treasury yield curve", *Journal of Financial Research*, 24, pp.545-564.

MASSOT, M. and NAVE, J.M. (2003), "La hipótesis de las expectativas a largo plazo: evidencia en el mercado español de deuda pública", *Investigaciones Económicas*, 27, pp. 533-564.

MASTRONIKOLA, K. (1991), "Yield curves for gilt-edged stocks: a new model. Bank of England", *Discussion Papers*, 49.

MCCULLOCH, J. (1975), " The tax-adjusted yield curve", *The Journal of Finance*, 30, pp. 811-830.

_ (1971), "Measuring the term structure of interest rates", *The Journal of Business*, pp. 19-31.

MICHAELSEN, J. (1963)," The term structure of interest rates", *Quaterly Journal of Economics*. 78, pp 166-174.

MODIGLIANI, F. and SUTCH, R. (1966), "Innovations in interest rate policy", *American Economic Review*, 56, pp.178-197.

MORENO, M. (2000), "Modelización de la estructura temporal de los tipos de interés: valoración de activos derivados y comportamiento empírico", *Revista Española de Financiación y Contabilidad*, 29, pp. 345-376.

MORINI, S. (1998), *Estimación de la estructura temporal de los tipos de interés: propuestas alternativas*.Unpublished PhD dissertation: Universidad de La Laguna.

NAVARRO, E. and NAVE, J. (1997), "A two-factor duration model for interest rate risk management" *Investigaciones económicas*, Vol. XXI, pp. 55-74.

NELSON, C. and SIEGEL, A. (1987), "Parsimonious modeling of yield Curves", *Journal of Busines,s* 60, pp. 473-489.

NUÑEZ, S. (1995), *Estimación de la estructura temporal de tipos de interés en España: elección entre métodos alternativo*s. Documento de trabajo Banco de España 9522, Banco de España.

PIAZZESI (2010), "Affine term structure models", en AÏT-SAHALIA, Y AND HANSEN, L (ed) *Handbook of Financial Econometrics*. Amsterdam: North Holland, pp. 691-766.

POWELL, M. (1981), *Approximation Theory and Methods*. Cambridge: Cambridge University Press.

RAMPONI, A. (2003), "Adaptative and monotone spline estimation of cross-sectional term structure", *International Journal of Theoretical and Applied Finance*, 6, pp. 195-212.
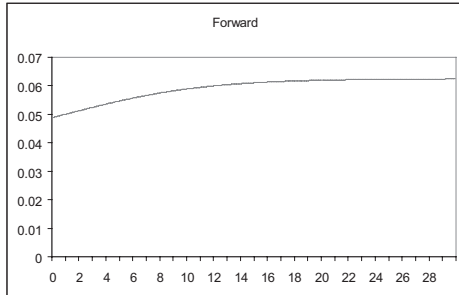
icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales,
nº 89 mayo-agosto 2013, ISSN: 1889-7045

151-154

SCHAEFER, S. (1981), "Measuring a tax-specific term structure of interest rates in the market for British government securities", *The Economic Journal*, 91, pp. 415-438.

_ (1979). *Consistent bond prices.*Unpublished PhD Thesis. University of London.

SHEA, G. (1984), "Pitfalls in smoothing interest rate term structure data: equilibrium models and spline approximations", *Journal of Finance and Quantitative Analysis*, 19, pp. 253-269.

SHILLER, R.J. (1990), "The term structure of interest rates", en FRIEDMAN, B AND HAN, F.H (ED) *Handbook of Monetary Economics*. Amsterdam: North Holland, pp. 626-723.

STEELEY, J. (1991), "Estimating the gilt-edged term structure: basis splines and confidence intervals", *Journal of Business Finance and Accounting*, 18, pp. 513-529.

SVENSSON, L. (1994), "Estimating and interpeting forward interest rates: Sweden 1992-1994". *Working Paper* No 114. International Monetary Fund

VASICEK, O. (1977), "An equilibrium characterization of the term structure", *Journal of Financial Economics*, 5, pp. 177-188.

VASICEK, O. and FONG, H. (1982), "Term structure modeling using exponential splines", *The Journal of Finance* , 37.

WAGGONER, D. (1997), "Spline methods for extracting Interest rate curves from coupon bond prices", *Working Paper* No 97-10. Federal Reserve Bank of Atlanta.

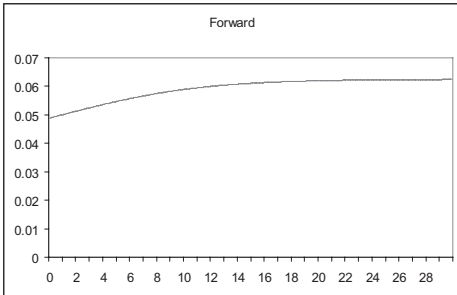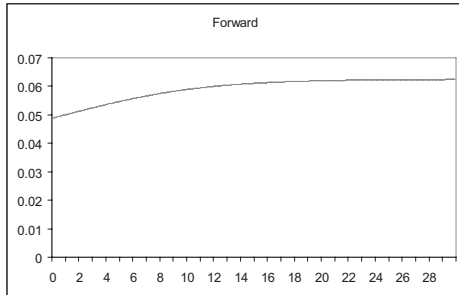## Annex 1. Shapes that the curve of interest rates may take

November 2002

December 2004

June 1996

January 1997

Source: Banco de España

icade. Revista cuatrimestral de las Facultades de Derecho y Ciencias Económicas y Empresariales, nº 89 mayo-agosto 2013, ISSN: 1889-7045

153-154

Annex 2. Estimation methods used by the principal central banks

Central Bank	Model
Belgium	Svensson/Nelson and Siegel
Canada	Exponential spline
Finland	Nelson and Siegel
France	Svensson/Nelson and Siegel
Germany	Svensson
Italy	Nelson and Siegel
Japan	Smoothed spline
Norway	Svensson
Spain	Svensson
Sweden	Smoothed spline/Svensson
Switzerland	Svensson
United Kingdom	VRP
USA	Smoothed spline

Source: Bank for International Settlements (2005)